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W. J. GREENSTREET, M.A., F.R.A.S.

A FIRST STATICS

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BY

Charles

C. S. JACKSON, M.A.

BARRISTER-AT-LAW ; LATE SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE ; INSTRUCTOR IN MATHEMATICS R. M. ACADEMY, WOOLWICH

AND

Robert

R. M. MILNE, M.A.

LATE SCHOLAR OF JESUS COLLEGE, CAMBRIDGE ; ASSISTANT MASTER ROYAL NAVAL COLLEGE, DARTMOUTH

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PREFACE

THE attitude of a student towards a new subject should be that of an investigator. This statement has been criticized because it has been misunderstood, but it is a truism. A student who is not wasting his time is exerting himself to find out something previously unknown to him. This is exactly what a discoverer is doing—with special ability and success.

A discoverer does not try random arrangements of a quantity of apparatus in the hope that "something may turn up." He tests a theory, hypothesis, or notion.

A student then should experiment to enlarge, confirm, clear up, and render exact his previous notion. Neither mere verification nor the determination of isolated facts is of much intellectual value. It is vain to attempt to defend the latter method—which the chemist calls "test-tubing"—by asserting that mechanics deals with things. All sciences deal with ideas and the communication of ideas. A bridge must exist in thought and must exist on paper before it can take form in steel.

On the other hand, it is generally agreed that to develop a science according to a strictly logical scheme from the smallest possible number of axioms is not a profitable course for a beginner. Ideas must be obtained before they

can be classified, and direct intuition sometimes furnishes a mental image which a deductive "proof" may only weaken.

There is, however, an intellectual satisfaction in realizing the unity and coherence of apparently distinct propositions, and our ultimate test of the truth of a theory or of a story is its perfect consistency with all its surroundings.

In trying to pay due respect to these principles we have been led—

(1) To consider the historical order of development of the subject, as indicating almost infallibly the line of least resistance.

(2) To lay stress on the practical utility of the science by showing its connection with machines of daily use.

(3) Not to lay great stress on rigour of deduction. Jumping to correct conclusions is not a serious intellectual fault.

We are much indebted to Professor Ewing, Director of Naval Education, for permission to make use of questions set at recent naval examinations, and to Professor C. Niven for similar permission as to questions set at the University of Aberdeen. The Controller of His Majesty's Stationery Office has allowed us to use some questions, marked Army, set at recent examinations.

We wish to acknowledge with gratitude the advantages we have derived from the ripe experience and great powers as a mathematician of Professor Hart, under whom we served for many years at the R.M. Academy, and from the generous aid which our friend Mr. W. M. Robert has given us in reading the proof-sheets.

We have, among other works, consulted the following, to which we may refer the student who desires to pursue the subject—Duhem, *Les Origines de la Statique;* Holtzmuller, *Die Ingenieur Mathematik;* Mach, *Die Mechanik in ihre Entwicklung* (English translation by T. J. McCormack); Minchin, *A Treatise on Statics;* Routh, *A Treatise on Analytical Statics.*

Messrs. Becker & Co. and Messrs. Dick, Kerr & Co. have kindly lent some blocks.

In the teachers' edition answers have been given to most of the examples: those omitted belonging to some of the questions asking for graphs, explanations, or tabulated results.

Information as to any misprints or errors which have escaped notice will be gratefully received.

C. S. J.
R. M. M.

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CHAPTER I

You have been learning facts about Mechanics nearly all your life. You know well that it would not do to use a tin-tack and a piece of cotton for hanging up a heavy picture. You know that two boys may be pulling their hardest, but that nevertheless if they pull opposite ways they will not do much towards moving a box. Such unconsciously acquired knowledge is most useful, but it is not precise. If, for instance, you wished to hang up a picture, you could only guess at the sort of nail and the sort of cord it would be best to use.

By the study of Mechanics we learn how to deal with many important problems: such as how to construct a bridge or roof to the best advantage; how to select a bar or rope which will be strong enough for the purpose for which it is intended, and yet not so excessively strong as to be needlessly costly and cumbrous; how to estimate the kind of engine needful for performing a certain task.

The science of Mechanics deals with the motion of bodies and with the circumstances which produce motion in bodies, or which permit a body to remain at rest. The branch of Mechanics which deals with bodies at rest is technically known as Statics.

Body.—We have spoken in the last paragraph of a body. This term is used to denote any definite object having weight. It will be noticed that this definition comprises not only complete objects in the ordinary sense of the word, but also any specified portion of such an object. Thus the body under consideration might be an object such as a book or a ladder, or a brick in a wall, or a link in a chain, or the top inch of a lead pencil.

Force.—A Force is the name given to any action which will cause a body to move, or which will stop it or turn it aside if it is already moving. If a body is let go and left quite unsupported, it will fall to the ground. According to our statement it is set in motion by some force, and this force is called the weight of the body.

We are quite accustomed to apply forces to various objects by muscular effort, and we have only to think over our experience to feel sure that a body which starts into motion from a state of rest always does so because some force has acted on it. If we saw a book which had been resting on the table suddenly rise slowly into the air, we should feel sure that in some way a force had been applied to it, and we should not be content until we found out "how it was done."

Though our notion of "force" is undoubtedly derived from our sensations of muscular effort, yet these sensations furnish us with but an imperfect means of comparing forces, just as we cannot rely on our sensation of heat to give us the temperature of a body. For instance, we cannot distinguish by our sensations between the weights of two

bodies which are nearly equal in weight. We may obtain means of comparing forces by observing the effects the forces produce on a given object. If each of two forces applied in turn to a spring distorts it by the same amount, and if we assume that the spring is in exactly the same condition when the second force is applied as it was when the first force was applied, we say that the forces are equal.

Spring Balance.—The use of a spring balance depends on the preceding principle. A spring balance consists simply of a spiral spring, furnished with a pointer and scale by means of which its elongation may be measured. A weight of 1 lb. hung from the spring produces a certain elongation. If we remove the weight and pull the spring until it has the same elongation we have exerted a force equal to the weight of 1 lb. The spring balance may thus be used for measuring forces, assuming that the spring remains in the same condition, notwithstanding its having been used several times.

It is clear that a force which, acting by itself on a body, would produce movement, may be neutralized by another force or other forces. Thus in a tug of war the knot in the centre of the rope is acted on by two very large forces, and yet as long as the struggle remains equal it does not move.

Experiment 1.

Attach two spring balances to a small ring by means of strings. Hook the end of one over a nail in a board and pull on the other. Are the two strings in a straight

line when the ring is at rest? Are the readings of the two balances the same?

The answers to these questions suggest an important principle, namely—If a body is at rest and two, and only two, forces are applied to it, which do not disturb it from its position of rest, they must be equal and act in opposite directions in the same straight line. The student may suggest that two men might try to pull the lid off a packing-case and yet not move it, though they were pulling in the *same* direction. In this case, however, additional forces come into play, namely, the resistance of the nails to be drawn out; and so the lid is not acted on by only two additional forces.

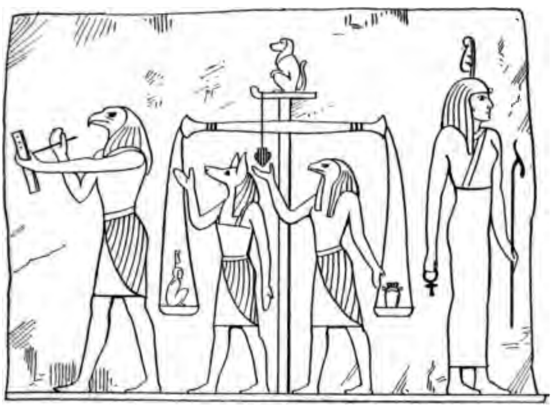
Action and Reaction.—When a weight is hung from a spring balance it pulls the spring down. The spring, on the other hand, prevents the weight from falling by applying an equal upward pull to it. This is an illustration of a more general result, which may be stated thus—Whenever two bodies exert force on one another the forces are equal in magnitude and opposite in direction.

This was enunciated by Newton in the words, "actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se semper esse æquales et in contrarias partes dirigi."—"To every action there is an equal and opposite reaction, or the actions between two bodies are always equal and directed opposite ways." Newton's own comment on this law is as follows (*Principia*, Motte's translation, ed. 1803, p. 15)—

"Whatever draws or presses another is as much drawn

or pressed by that other. If you press a stone with your finger the finger is also pressed by the stone. If a horse draws a stone tied to a rope the horse, if I may so say, will be equally drawn back towards the stone, for the distended rope by the same effort to relax itself will draw the horse as much towards the stone as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other."

The Balance.—Every one is familiar with the use of a pair of scales. Indeed the balance may be described as the oldest and the most universally known of scientific instruments. Among the Egyptians, many thousands of



years ago the balance was already a symbol of justice and retribution. The figure, from the British Museum, represents the balance in which the soul of the dead man was supposed to be weighed in the Hall of the Dead. Our first object will be then to study the principle of the balance.

Experiment 2.

A ruler[1] can turn about an axis above its centre. The ends of the axis rest on supports, which are at the same level. The ruler will take up a position of rest, about

which it should oscillate if slightly disturbed. It is well to see that it does so oscillate, so that we may be sure that it is free to turn about the axis. If we attach equal weights, say ½ lb. each, at equal distances on each side of the centre, it will be found that the ruler will come to rest in its former position. The experiment should be repeated, using different weights. We conclude that equal weights hung at equal distances from the axis of a lever will not disturb it from its position of rest.

Experiment 3.

The question immediately arises what would be the effect of hanging weights at unequal distances from the axis of a lever. Hang a weight, say of 4 lbs., at 9 inches from the axis or *fulcrum*. Find by trial what weight hung at distances of 1, 2, 3, 4, 6, 9 inches on the other side will maintain the lever in its horizontal position of rest. Notice that if a slight addition is made to the balancing weight on one side, the ruler begins to move, and will no

[1] There is no difficulty in constructing this piece of apparatus with an ordinary 12-inch wooden ruler and a knitting-needle as axis. It can, however, be purchased from Messrs. Cusson, of Manchester, Messrs. Becker, and no doubt elsewhere.

longer remain in a horizontal position. Draw up a table of weights in the following form—

Distance from fulcrum.	Weight keeping the lever horizontal.

You will probably conclude that the law may be expressed thus—

IF A HORIZONTAL LEVER BALANCES ABOUT ITS FULCRUM AND WEIGHTS ARE ATTACHED TO IT ONE ON EACH SIDE, SO THAT IT STILL BALANCES, THEN THE WEIGHT ON ONE SIDE MULTIPLIED BY ITS HORIZONTAL DISTANCE FROM THE FULCRUM IS EQUAL TO THE WEIGHT ON THE OTHER SIDE MULTIPLIED BY ITS HORIZONTAL DISTANCE FROM THE FULCRUM.

We may express the conclusion by a formula. If under the same circumstances as before a weight P at a distance a from the fulcrum balances a weight Q at a distance b from the fulcrum, then $Pa = Qb$.

EXPERIMENT 4.

We shall now endeavour to test more closely the law suggested by the previous experiment. The beam of the balance (*i.e.* the ruler) is to be kept horizontal by means of a spring balance, with one end fixed to the ground and the other to one end of the beam, say, 9 inches from the fulcrum. If the ½-lb. weight is now suspended from the other arm at distances 1, 2, 3, 4, etc. inches from the fulcrum, the weight at the other end required to keep

the balance at rest may be read off directly on the spring balance. The results should be entered in a table thus—

Distance of weight from fulcrum.	Reading of spring balance.	Reading we should expect to obtain, judging from the preceding experiment.

Should you expect to obtain the same reading from a given pull on the spring balance, whether the balance is upside down or erect? Can you see how to avoid any error on this account in the experiment?

Use of Squared Paper.—We may adopt a different method of exhibiting the information contained in the table. Taking some squared paper, set off from a fixed point horizontal distances equal to those in the first column of the table, and at each point reached measure off a vertical distance proportional to the corresponding reading of the spring balance. The student must use his common-sense here and act as if his diagram were to be framed and hung up. It would be absurd to frame a large sheet of paper with a small picture or diagram in one corner of it.

The student should first notice the range of quantities to be represented. Here the horizontal distances vary from 1 to 9 inches, while the vertical distances have to represent weights, which vary from $\frac{1}{2}$ to $4\frac{1}{2}$ lbs. He should next note the size of the sheet of paper available for the diagram, and choose to represent the quantities

involved such distances as will cause the diagram to fit easily on the sheet. Thus, suppose that in the present case the sheet of paper is 10 inches square. We might conveniently represent 1 inch by an inch measured horizontally, and 1 lb. by 2 inches measured vertically. The student must not forget that a simple relation between the quantity represented and the distance representing it is desirable. He must be careful to write on the diagram a clear statement of what the vertical distances represent, and what the horizontal distances represent.

Each line in the table will furnish a point in the diagram. Do you notice any simple geometrical condition which these points appear to fulfil? You will probably observe that all the points nearly lie on a straight line. Should you expect that, if we took further observations with the apparatus and plotted the points corresponding to them, these new points would fall close to the same straight line? Try. You should now draw the straight line which you think is most clearly indicated by the position of these points. What use can you make of this straight line when you have drawn it? Mark a point on the straight line. What readings in your table of results would this point have represented? You should now try to verify these readings.

Experiment 5.

Pressure on Fulcrum.—Returning to our original experiment—2—in which we arranged two weights keeping the balance horizontal, we notice that the lever serves the purpose of suspending the weights from the fulcrum.

It naturally occurs to us to consider whether the total force on the fulcrum depends on the position of the weights. This may be easily tested by suspending the lever from a spring balance in place of the support previously employed. The tension or pull of the string will be indicated by the reading of the spring balance. Take a series of readings for various positions and weights, such as those mentioned on page 8, and enter them in a table in the following manner—

	Sum of weights.	Readings of balance.
1.		
2.		
3.		
4.		

Are the numbers in the two columns equal or nearly so? To what cause do you attribute any difference? What is the weight of the lever itself? You will probably conclude that the difference of the results in the two columns is due to the weight of the lever, and if you have used a lever of appreciable weight, the reading of the spring balance will indicate that the force on the fulcrum is equal to the sum of the weight of the lever and the weights suspended from it. In other words, the force exerted by the lever on the fulcrum is a downward pull equal to the sum of the weights, while the force exerted by the fulcrum on the lever is an upward pull equal to the sum of the weights.

Resultant.—We perceive that as far as the effect on the fulcrum is concerned, a single force, $P + Q$, applied at O is exactly equivalent to the separate forces P and Q (see figure, p. 19). The force $P + Q$ at O is said to be the *resultant* of P at A and Q at B. Observe that P and Q have a tendency to bend the lever round O, which the resultant would not have. If, therefore, we desire to investigate the bending of the lever we cannot replace the two forces by their resultant. Notice that when the points A and B are given the position of O can be found. A few simple numerical examples will make this clear.

Examples.

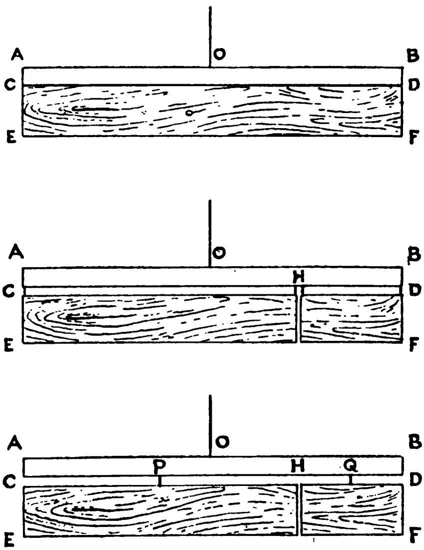
1. A stick 5 feet long balances about a point 2 feet from one end. If a weight of 60 lbs. is hung from that end, what weight is required at the other end to balance it? What is the pressure on the fulcrum?

2. A pencil 7 inches long has weights of 1 lb. and $\frac{3}{4}$ lb. attached to its ends. At what point must it be supported for these weights to balance? (We shall see subsequently how to allow for the weight of the pencil, but as its weight is only about $\frac{1}{4}$ oz., we naturally anticipate that its effect on the position of the fulcrum will be small, so neglect it for the present.)

3. Two men carry a weight by means of a horizontal pole 7 feet long. If the weight = 1 cwt., and if it is attached to a point 3 feet from one end, find how the weight is distributed between the two men. (Neglect the weight of the pole.)

The two conclusions at which we have arrived, (1) $Pa = Qb$, and (2) pressure on the fulcrum $= P + Q$, have been deduced from the results of experiment. Experimental results, however, do not agree exactly, but only approximately with these laws. Attempts have been made from the earliest times to deduce these laws by a process of strict mathematical reasoning from

axioms, which will be accepted by every one. The axiom which has been usually adopted is that a uniform heavy rod will balance about its middle point. This was looked upon not as an experimental truth but as obvious apart from experiment. From this axiom the principle of the lever was deduced by Archimedes (287–212 B.C.). It was afterwards employed in a somewhat different manner by Stevinus of Bruges (1548–1642 A.D.), and by Galileo (1564–1642 A.D.). Following Prof. Mach (*Science of Mechanics*, p. 12) a sketch of the proof is here given.



The figure represents a straight uniform rod,[1] AB,

[1] If we prepared a number of rods, making each as nearly straight as possible, as nearly the same thickness as possible, and as nearly uniform (that is, as nearly the same weight per inch) as possible, we should find that each balanced about a point, which was very

suspended from its middle point O, with a second uniform heavy beam, CDEF, attached to it. The axiom tells us that this will balance about O. If we divide the beam into two portions by a vertical cut at H, and tie these two portions to the rod at their extremities, we have done nothing to change the state of rest of the system. Each of the two portions of the beam may now be suspended from the bar by a single thread at their middle points (axiom). Let CH be m units in length and HD be n units, so that the length of the beam or rod is $m + n$. Then, since $AO = \frac{1}{2}(m + n)$ and $CP = \frac{1}{2} m$, therefore the horizontal distance of P from O is $\frac{1}{2}n$, and in like manner the horizontal distance of Q from O is $\frac{1}{2}m$. If each unit of length of the beam weighs 1 lb., the weights of the two portions are m and n lbs. respectively. Therefore a weight of m lbs. at a distance of $\frac{1}{2}n$ from the fulcrum balances a weight of n lbs. at a distance of $\frac{1}{2}m$ from the fulcrum. Further, the two portions weighing m and n lbs. respectively make up the whole beam CDEF weighing $m + n$ lbs.

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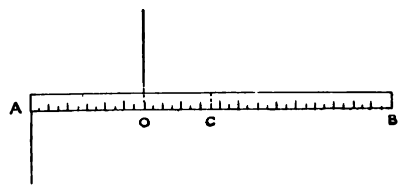
near to the middle point. More than this we cannot say, for we cannot make a rod exactly straight, nor can we tell the exact middle point. In geometry the student has already met with the same difficulty. If we defined a circle as a figure drawn on a flat piece of paper with a pair of compasses, we should have to say that the angle in a semi-circle is very nearly a right angle, and that the better our instruments and the more careful our work, the more nearly would the angle be a right angle. In geometry all these difficulties are got rid of by reasoning about the ideal circle. So in the following proof we reason about an ideal lever perfectly straight and perfectly uniform, which would exactly balance about its middle point, and the properties which the ideal lever possesses exactly are nearly possessed by the levers which we use.

Hence the resultant of m lbs. at P and n lbs. at Q is $m + n$ lbs. at O.

Centre of Gravity of Uniform Heavy Lever.—We commenced our previous experiments by ascertaining a position for the fulcrum about which the lever would balance, without weights attached. We shall now proceed to examine how to make the lever balance about any point.

EXPERIMENT 6.

Fix the support at a known distance from the end A of a heavy rectangular metal rod, and denote the point of support by O. (Flat bar iron or brass can be obtained in various sizes, a piece 18 in. × 1 in. × ¼ in., weighing about 1¼ lbs., would be suitable. A narrow strip of squared paper, to serve as a scale, can be gummed along it.) First find the



weight of the lever itself, then using the spring balance find what pull at A will hold the lever in a horizontal position. What, in your opinion, is the force which, acting on the other side of the lever, balances the pull at A? Measure or read off on the scale the length AO. Compute at what point, C, a force equal to the weight of the lever should act, in order to balance the pull at A? What

according to this computation is the length of AC? If you repeat this experiment for varying positions of O, you should find that AC is the same length in every case. The bar is probably very uniform in breadth and thickness, and if so the point C will be very nearly at its centre. This experiment suggests that, for the purpose of balancing a uniform straight lever, its weight may be regarded as a force acting at its centre.

Centre of Gravity.—Anticipating the results of subsequent work, we may here state that, in the case of every body, there is a point termed its Centre of Gravity, through which its weight may be supposed to act when we are considering the forces acting on it as a whole. If the body is supported by an axis, which passes through it centre of gravity, the body will rest in any position.

Illustrations and Examples.

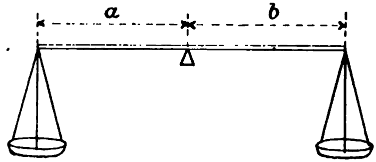
The applications of the principle of the lever in ordinary daily life are almost innumerable. The student should try to discover additional illustrations for himself.

Examples.

4. What do you think are the essentials of a fair set of weights and scales? (Weights correct, arms equal, balances when no weights are on, turns freely.)

5. The correct weight of a body can be found even if the balance has unequal arms. Put the body to be weighed into one scale-pan and counterbalance it with shot or sand, then remove the body and in its place put weights until the shot or sand is again balanced. The sum of these weights gives the weight of the body.

6. Another mode of obtaining the correct weight of a body, using a balance having unequal arms, is as follows—

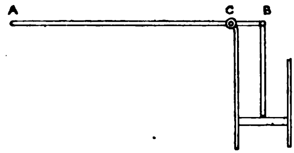


Denote the true unknown weight of the body by x, and let b and a be the lengths of the right- and left-hand arms of the balance respectively. Place the body in the left-hand scale-pan and place weights in the other pan till it is balanced. Let these weights amount to W_1 lbs. By the principle of the lever we have $xa = W_1b$, and therefore $x = \frac{W_1b}{a}$. If we could measure a and b accurately or find their ratio, this equation would give us x. But in a balance as ordinarily constructed the direct measurement would be extremely difficult. The ratio may be determined by putting a known weight W_2 into the left-hand scale-pan and balancing it with known weights (sum $= W_3$) in the other pan, then $W_2a = W_3b$, and therefore $\frac{x}{W_2} = \frac{W_1}{W_3}$.

Double Weighing.—In practice, without actually determining the ratio a to b, we can get rid of it by placing the body in the right-hand scale-pan and balancing with known weights, W_2, in the other scale-pan. This leads to the equation $W_2a = xb$. Combining this equation with the previous one, namely $xa = W_1b$, we get $\frac{x}{W_2} = \frac{W_1}{x}$, or $x^2 = W_1W_2$, or $x = \sqrt{W_1W_2}$. This method is known as the method of double weighing, and is sometimes employed when great accuracy is required.

7. A body appears to weigh 20 grms. when placed in one scale-pan of a balance and 21 grms. when placed in the other scale-pan. What is the true weight of a body which appears to weigh 10 grms. when placed in the first scale-pan, supposing the difference due to unequal length of the arms of the balance?

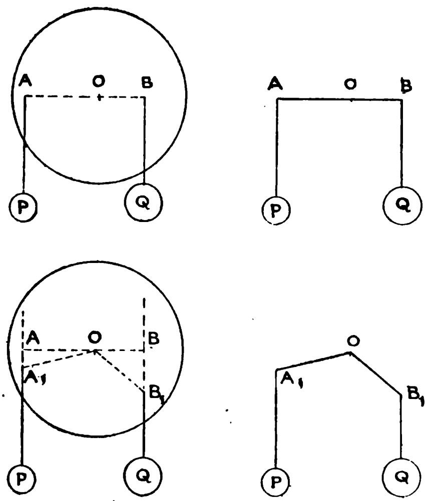
8. The handle of a pump (see figure) is a rod, AB, 4 feet long, pivoted so as to turn about a point C, 6 inches from B. If a downward force of 25 lbs. is applied at A, what is the upward pull exerted on the piston-rod at B?



9. A contrivance for raising and moving buckets of water, used in China, Egypt and India, consists of a long pole, AB (say 16 feet long), pivoted at a point C (say 3 feet from A), and weighted at A so as to balance about C. The bucket is attached at B and the workman lifts it by pushing A downwards. If the bucket weighs 16 lbs., what force must the workman exert to raise it?

A similar contrivance is used by fishermen in various parts of the world for raising and lowering a net.

Lever replaced by Circular Disc.—We have hitherto dealt



with straight horizontal levers. Let us consider a uniform circular disc, which can turn freely in a vertical plane about

its centre O. Suppose two small drawing-pins, inserted at points A and B on a horizontal diameter, and weights P and Q hung from them, to balance one another. Comparing this case with that of a straight lever AB, which balances at O, it is at once apparent that the relation between P and Q is the same as if the disc were replaced by this straight lever, and therefore $Pa = Qb$, where $OA = a$, $OB = b$. Draw vertical lines on the disc through A and B and put drawing-pins at any two points A_1 and B_1 on these lines respectively. It appears evident that if P and Q were hung from A_1 and B_1 instead of from A and B, the disc would still be at rest; and, further, that the relation between P and Q is the same as if they were hung from a bent lever A_1OB_1, which balanced about O. We can now enunciate the principle of the lever in a form that is applicable to a lever of any shape.

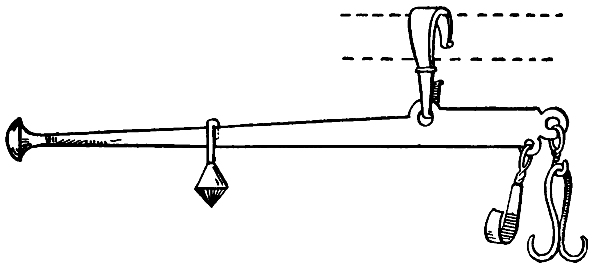
Two vertical forces, P and W, applied to a lever tending to turn it in opposite directions will balance one another, if the product of P and the length of the perpendicular from the fulcrum O on the line of action of P is equal to the product of Q, and the length of the perpendicular from the fulcrum O on the line of action of Q.

Moment of a Force about a Point.—To shorten this lengthy statement a technical term is introduced. The product "P multiplied by the length of the perpendicular from the fulcrum O on the line of action of P" is termed "the *moment* of P about O," and so the principle may be stated thus—

Two vertical forces, P and Q, applied to a lever tending

to turn it in opposite directions will balance, if their moments about the fulcrum are equal.

The Roman Steelyard.—This consisted of a heavy lever hung up from a fixed point. The object to be weighed was hung from the short arm, and balanced by a known weight (*e. g.* 10 lbs.), which could be moved along the long arm until equilibrium was attained. The longer arm was



graduated so that the weight of the object could be read off from the position at which the movable weight maintained equilibrium. The figure represents an ancient Roman instrument of bronze found in Gloucestershire, dating from the Roman occupation of Britain.

Examples.

[In Examples 10–14 the lever is uniform, the shorter arm $\frac{1}{2}$ foot long and the longer arm 2 feet long.]

10. What should be the weight of the scale-pan in order that the lever, whose weight is 5 lbs., may balance when no load is on, and movable weight is off?

11. What should be the weight of the scale-pan if the lever is to balance with no load on, when the movable weight of 16 lbs. is 2 inches from B?

12. What is the greatest load which the steelyard in the preceding question could weigh?

13. Find the position of the marks, indicating the position of the movable weight, for loads weighing 10, 15 and 20 lbs.

14. Draw a graph showing the relation between the weight of the load and the distance of the movable weight from the fulcrum, and hence show how to graduate the instrument so as to indicate the weight of the load in pounds.

15. Given the length and weight of the lever and the weight of the scale-pan, how would you find from what point to suspend it so that it would remain horizontal?

16. Suppose a steelyard found in good condition, but without a movable weight, how would you find the proper movable weight to use with it?

17. A tradesman employs a Roman steelyard for weighing his goods. The travelling weight is 4 ozs., the knife-edge of the scale-pan is 5 inches from the fulcrum, and the zero mark is 2 inches from the fulcrum, on the opposite side from the scale-pan. Having lost the travelling weight he employs in its place one of 8 ozs., multiplying the readings of the instrument by two. Find whether he, or his customers, gains by the change, and by how much.

18. A light rod, 12 inches long (whose weight may be neglected), is suspended in a horizontal position by two vertical strings, each of which can just sustain 100 lbs. What is the greatest weight that can be hung from (1) the centre of the rod, (2) a point 2 inches from the centre, without breaking a string? Assume the strings to remain vertical and the rod to remain horizontal.

19. Leonardo da Vinci proposes the following question (*Duhem*, p. 157)—

"To try a man, and see whether he really understands the nature of weights, ask him at what point one should cut one of the equal arms of a lever, in order that the piece cut off, tied on to the end of the remainder, may exactly balance the opposite arm. If he tells you how much to cut off he is no mathematician." Examine this.

20. A man applies a vertical downward force P to a handle, whereby he raises a weight W by a rope which coils up on a horizontal cylinder.

Calculate the magnitude of P when the angle which the handle makes with the horizontal is 0°, 20°, 40°, 60°, 80°, taking W = 100 lbs., length of handle 24 inches, radius of cylinder 4 inches.

What would happen if the handle were vertical?

Work.—When a man raises a weight through a height, he is said in technical language to do work. The amount of work done is measured by the product of the weight and the vertical height through which it is lifted. To lift 1 lb. through a vertical height of 1 foot requires 1 foot-pound of work. Hence to lift W lbs. 1 foot requires W foot-pounds of work, and to lift W lbs. h feet requires Wh foot-pounds of work. Whenever a word is used both in a popular language and in a technical sense, particular care should be taken that the technical meaning of the word is precisely understood.

In the first place, "work" in the technical sense has no reference to time. Two men who each carried a hundred-weight of coal up a flight of stairs, the one in two minutes and the other in five minutes, have each done the same amount of work in lifting the coal, or, as we may say, on the coal. Further, no work is done on the coal by merely supporting it without raising it. It may fatigue the man to have to stand still, with the sack on his back, and in a popular sense this may be part of his day's work, and in a physiological sense his muscular or nervous system may have done work, but it is not work done on the coal in the mechanical sense of the word.

Example.

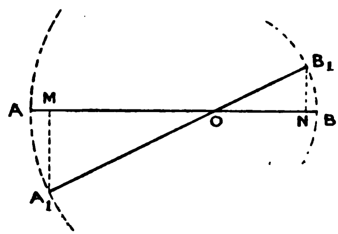
21. How much work must be expended in raising a lift, weighing 16 cwts., through a height of 90 feet, putting any waste out of the question? Specify carefully the units you adopt.

We shall now consider whether any diminution in the

work necessary to raise a weight can be effected by using a lever.

To simplify matters we will assume, either that the centre of gravity of the lever is exactly at the fulcrum or else that the weight of the lever is so small, in comparison with the other forces acting, that we may leave it out of account.

Suppose AB a straight horizontal lever whose fulcrum is at O. Let $AO = a$ and $BO = b$. Suppose a weight Q at B is balanced by a vertical force P at A. We know that $Pa = Qb$. Let the lever be brought to the position A_1OB_1; the weights P and Q will still balance one another, for the



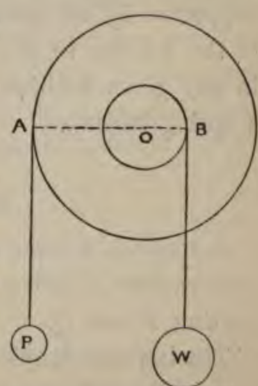
condition that they should do so is that $P \times OM = Q \times ON$, A_1M and B_1N being perpendicular to AB. But we know that $P \times OA = Q \times OB$, also that $\dfrac{OM}{OA_1} = \dfrac{ON}{OB_1}$. Multiplying the corresponding sides of these two equations together, we have $\dfrac{P \times OA \times OM}{OA_1} = \dfrac{Q \times OB \times ON}{OB_1}$, or $P \times OM = Q \times ON$.

Since the forces P and Q balance at every instant as the lever is turned from the position AOB to the position A_1OB_1, it follows that a force slightly greater than P would cause the movement from the one position to the other to

take place. The vertical height through which Q has risen is B_1N, the vertical height through which P has fallen is A_1M. But $\frac{A_1M}{OM} = \frac{B_1N}{ON}$. Multiplying the corresponding sides of the last two equations together we have: $P \times A_1M = Q \times B_1M$. That is, P multiplied by the vertical distance through which it has moved = Q multiplied by the vertical distance through which it has moved.

We see then that although P may be very much smaller than Q, and thus a great weight Q may be raised by means of a small force P, yet the work necessary to raise Q through a given height is the same with or without the lever. This result is an illustration of the maxim that "what is gained in power is lost in motion." An example will make this more clear.

Wheel and Axle.—A weight can only be raised a small distance by an ordinary lever, without constantly shifting the fulcrum. If the weight is to be raised a considerable height a contrivance called a wheel and axle or windlass may be employed.



A horizontal barrel or cylinder of small radius can turn about its axis. A large wheel is fixed to the barrel and can turn about the same axis. A rope coiled on the barrel supports the load W, while the lifting force is applied at the rim of the wheel in a tangential direction, by means of a handle. For

experimental purposes we may employ a cord coiled on the rim of the wheel to which a weight is attached. As the rope coils up on the barrel the cord uncoils from the rim, and we see that at every instant the apparatus furnishes a horizontal lever AOB, and that P will consequently balance W if $P \times OA = W \times OB$.

Let the radius of the wheel = 18 inches,
" " " barrel = 4 "
Let W = weight = 40 lbs.

Find the force P that will balance the weight. A force somewhat greater than your answer will raise W. How far must the weight P descend in order to raise the weight W 5 feet?

Fulcrum at one End.—In the cases hitherto considered the forces P and Q have been on opposite sides of the fulcrum. It is clear, however, that the weight Q could be supported by an upward force applied on the same side of the fulcrum as Q. The force P required to support Q in any given position could be found experimentally by a spring balance.

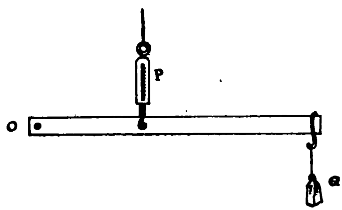
Experiment 7.

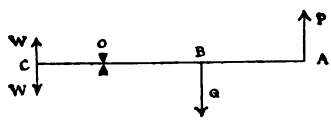
Arranging a lever as in the figure, take a series of observations of values of P and Q and state the law connecting P and Q.

The law might, however, be deduced from the preceding case in the following manner—

Let downward pulls W at C and Q at B balance on

the lever as in the figure, then we know that $W \times OC = Q \times OB$. Let upward pulls P at A and W at C balance one another on the same lever, then $P \times OA = W \times OC$, and it is evident that the four forces applied simultaneously

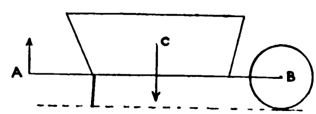




would balance one another. The pulls at C balance one another and may be removed, and thus P at A will balance Q at B, provided $P \times OA = Q \times OB$.

Examples.

22. **Wheelbarrow.**—In the figure A is the handle, B the axle of the wheel, and the weight may be supposed always to act in a vertical line through C. When the barrow is on the ground AB is horizontal, and C 8 inches above the middle point of AB.



If AB is 40 inches and W = 160 lbs., find the vertical force at A necessary to just raise the barrow.

What vertical force is necessary to hold the barrow with AB inclined to the horizontal at 10°, 20°?

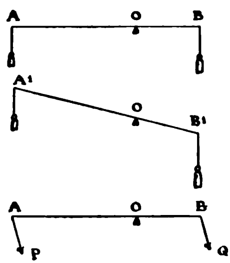
23. **Wire-nippers.**—Suppose the wire to be $\frac{1}{4}$ inch from the

fulcrum and the pressure to be applied to the handles 8 inche from the fulcrum, what force must be exerted on each handle to produce pressures of 200 lbs. on each cutting edge?

What reaction must the fulcrum exert against each limb of the nippers?

Any Parallel Forces.—We have hitherto considered the lever as subjected to *vertical* forces. The restriction to vertical forces may, however, be removed in the following manner. To simplify the argument we shall consider a lever whose weight may be left out of account.

We have already noticed that if a lever balances about O, in the position AOB when weights P and Q are attached to it, it will still balance after being turned through any angle into the position A_1OB_1 (see figure). The balancing of the forces depends only on their magnitudes and positions relative to the lever. If we had supposed the directions of the forces (figure) to be turned through the angle a, the same positions of the forces relative to the lever would have been obtained.



It appears then that the condition which we have proved for vertical forces is true for any case of parallel forces, and we realize that the pressure on the fulcrum is parallel to the directions of the given forces. Considering the forces that keep the lever at rest or in equilibrium, we have the two given forces P and Q and the pressure of the fulcrum on the lever. This pressure is equal in

magnitude and opposite in direction to the force exerted by the lever on the fulcrum, which we have called the resultant of the forces P and Q. It is useful to have a term to denote the force which is equal and opposite to the resultant of P and Q, and, as this force together with P and Q keeps or leaves the lever at rest, it is called the *equilibrant* of P and Q.

We have already observed cases in which a body, on which forces acted, might be considered as equivalent to a lever. Thus, if three parallel forces keep a body in equilibrium, they will keep the hypothetical lever in equilibrium and will satisfy the following conditions.

Equilibrium of three Parallel Forces acting on a Body.— If a body is in equilibrium under three parallel forces P, Q and R acting at three points A, B and O in a straight line, the middle force R acting at O must act in the opposite direction to the other two forces and must be equal to their sum; furthermore, the forces P and Q must fulfil the condition that $P \times OA = Q \times OB$.

The last two conditions may be concisely expressed by the equations $\frac{P}{OB} = \frac{Q}{OA} = \frac{R}{AB}$, or, in words, that each force is proportional to the distance between the other two. The advantage of these equations is that they enable us to deal with all cases in which three parallel forces maintain a body in equilibrium. It is frequently said that in this case the three forces are in equilibrium. To speak with strict accuracy, we ought to say that three forces which satisfy the above equations do not disturb

the equilibrium of the body on which they act. It is incorrect to say that they maintain it in equilibrium, for this implies that if they ceased to act the body would no longer be in equilibrium; it is still more incorrect to say that they are in equilibrium of themselves. These two phrases, however, are convenient for their brevity and are universally employed.

We have already stated that if three parallel forces, P, Q, R, satisfy the equations $\frac{P}{OB} = \frac{Q}{OA} = \frac{R}{AB}$, a force equal and opposite to R is the resultant of P and Q. It is equally the case that the resultant of any two of the forces is a force equal and opposite to the third. It is usual to speak of parallel forces which have the same direction as like parallel forces, and those which are in opposite directions as unlike.

Examples.

24. Find the magnitude of the resultant, its distance from A, and its direction in the following cases—

Like Forces.

(a) $P = 10, Q = 1,$ $AB = 11$ inches.
(b) $P = 10, Q = 10,$ $AB = 11$ „
(c) $P = 10, Q = 100,$ $AB = 11$ „

25. Two like parallel forces of magnitudes 12 lbs. and 9 lbs. act in lines 3 feet apart. Find their resultant in magnitude, position and direction.

26. Two unlike parallel forces of magnitude 7 lbs. and 5 lbs. have their lines of action 3 feet apart. Find their resultant in magnitude and draw a diagram showing its position and direction (scale, 1 inch to 1 foot).

27. Two like parallel forces of 5 lbs. and 4 lbs. act on a body at a distance of 18 inches apart. To what single force are they equivalent?

28. A cutting knife, ABC, turns about a pivot at one end, A, it is pressed downwards by a handle at B, and cuts an object at C. If $AB = 10$ inches, $AC = 4$ inches, what pressure is exerted on the object at C by a downward force of 5 lbs. at B?

29. A uniform log 6 feet long weighing 4 cwts. rests on supports 2 feet from one end and 6 inches from the other end. What is the pressure on each support?

[The student is recommended to draw a figure to scale. The centre of the log is 1 foot from one support and $2\frac{1}{2}$ feet from the other support.

If O is centre of the log, A the left-hand support and B the right-hand support, then $AO = 1$ foot, $OB = 2\frac{1}{2}$ feet. If P and Q are the pressures on the supports, then—

$$\frac{P}{OB} = \frac{Q}{OA} = \frac{R}{AB},$$

$$\frac{P}{2\frac{1}{2}} = \frac{Q}{1} = \frac{4 \text{ cwts.}}{3\frac{1}{2}}$$

$\therefore P = 2\frac{6}{7}$ cwts. and $Q = 1\frac{1}{7}$ cwts.]

30. A straight uniform lever 5 inches long weighing 3 lbs. can turn freely about one end, and is in equilibrium when acted on by two parallel vertical forces of 7 lbs. and 5 lbs. respectively. The force of 7 lbs. acts vertically upwards at the free end. Find the point of action of the force of 5 lbs.

31. A uniform rail, ABCD, 20 feet long, weighing 56 lbs. per foot, rests on a prop at C, and is pressed downwards by a prop at B. If $AB = 3$ feet and $BC = 4$ feet, and the rail ABCD is horizontal, find the pressures at B and C.

32. A uniform plank, ABC, of length 12 feet and weight 80 lbs. rests on two supports at A and B, one at the end A and the other $4\frac{1}{2}$ feet from the end C. A boy walks along the plank from A to C, and just as he reaches C the plank commences to tilt. Find the weight of the boy.

33. A bar, LMN, rests in a horizontal position in contact with two fixed pegs M, N, and a weight of 20 lbs. hangs from the end L. If LM be 13 inches and MN be 8 inches, find the reactions of the pegs. Determine also the change in their values if the weight be moved x inches nearer to M.

34. Two men carry a weight of 1 cwt. on a bar whose length is

8 feet and which weighs 7 lbs., an end of the bar resting on the shoulder of each man. Where must the weight be placed on the bar if the pressures supported by the men are in the ratio 8 : 9?

35. The horizontal roadway of a bridge weighing 8 tons is 12 yards long, and rests on similar supports at its two ends. What pressure is exerted on each of its supports when a cart weighing 2 tons is one-quarter of the way across?

36. A uniform horizontal girder weighing 21 cwts. rests symmetrically on supports whose centres are 20 feet apart. What will be the pressures on the supports when there is an additional load of 20 cwts. at 8 feet from the centre of the left-hand support of the girder.

37. A uniform beam of oak 10 feet long, weighing 240 lbs., is to be carried by three men, one at an end, the other two each taking one end of a short stick passed under the beam. How must the stick be placed if each man is to support one-third of the load?

[Consider the resultant force of the two men as acting on the beam.]

Is there any common-sense reason against putting one man at each end of the beam and one at the centre?

CHAPTER II

IMAGINE two boys, pulling each with a force of 50 lbs. trying to move a big roller by means of two ropes.

If the two ropes are not parallel it is clear that the combined efforts of the boys are not equivalent to a single force of 100 lbs. It is also clear that there is some single pull which would have the same effect towards moving the roller as the two pulls of 50 lbs. We should like to find this single force.

Resultant.—The problem may be stated in a more definite and abstract form as follows. Two forces of given magnitudes are applied at a given point of a body in given directions. Find their *resultant*, that is to say, find the single force which, applied to the given point, will produce the same effect on the body as the two given forces.

A force equal and opposite to the resultant and acting in the same line would neutralize it, and would consequently neutralize the original forces. As in the case of parallel forces, this force equal and opposite to the resultant is called the *equilibrant* of the two original forces.

The student may suggest that the two forces which the boys exert might not be applied to the same point of the roller. This is true, and we shall consider the more

general problem of two forces applied at different points of a body subsequently.

An expression in the statement of the problem seems to require explanation. We have spoken of the forces as applied "at a point." In geometry any straight line that we can draw possesses breadth, but we reason about an ideal straight line which has no breadth, and find that our conclusions are, to a certain extent, correctly realized in the figures that we draw. The forces which we are considering are applied to small areas or portions of the surface of a body. In certain cases these areas are very small, as for instance the surface of contact of a piece of cotton with a nail to which it is tied. In such cases we reason about ideal forces applied at a geometrical point, and we find that our conclusions are to a certain extent correctly realised in the objects with which we deal.

Experiment 1.

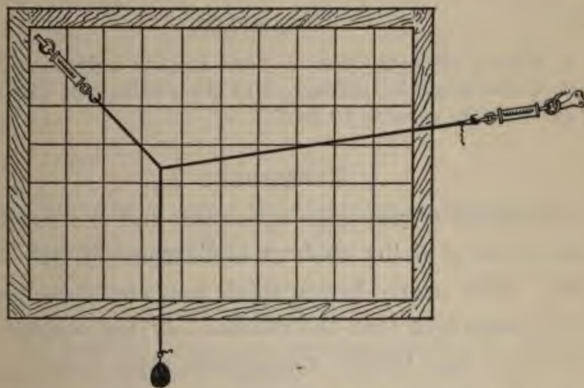
Two hooks are screwed into the top corners of a vertical blackboard. Spring balances are slipped over these hooks and threads attached to them knotted to a thread to which a weight R, say of 10 lbs.,[1] is attached.

Slipping behind the threads a sheet of paper fixed to a drawing-board the lines of the threads can be marked on the paper by pricking through at two or three points on each thread as far apart as possible. If you prick the paper at the *side* of the thread, instead of through it, is any error necessarily caused? It is as well to prick through three points on each thread. The line through

[1] The weight suitable depends on the kind of spring balances used.

any two of these should pass through the third, and a test of the accuracy of the work is thus obtained. A further check is that the three lines should meet in a point.

Take the readings of the spring balances. Let these be "P" and "Q." Measure the angles the threads make



with one another with a protractor and record the result of your experiment thus—

The equilibrant of "P" and "Q" at an inclination "   " is a force of "R," 10 lbs., in a direction making an angle "   " with the direction of "P."

The resultant of "P" and "Q" at an inclination "   " is a force of "R" in a direction making an angle "   " with the direction of "P."

Note that if you have measured the three angles at the knot independently you can obtain a further check of the correctness of your work by adding up the three angles.

Keep the results of the experiment for subsequent use.

This experiment will render it clear that the resultant

of two forces of 6 and 4 lbs. is not in general a force of 10 lbs., and further experiments of the same sort which the student can arrange himself will indicate that the size of the resultant depends on the inclination as well as on the size of the two original forces.

EXAMPLES.

1. What is the equilibrant of P and R, at an inclination AOC?
2. Under what circumstances will the resultant of two forces of 4 and 6 lbs. be equal to 10 lbs.?

EXPERIMENT 2.

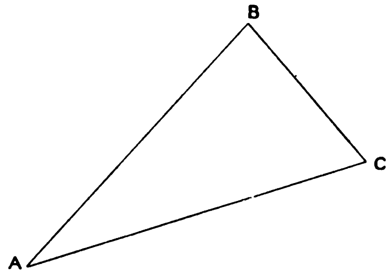
By means of the same apparatus find the resultant of two forces of 5 lbs. each at inclinations of 60°, 90° and 120°. How nearly do you think your angles are correct?

We have seen that the resultant, or the equilibrant, of two forces may thus, in manageable cases, be found experimentally. Our next problem is: Given the magnitude and direction of two forces acting at a point, to predict, without experimenting, the magnitude and direction of their resultant. All we have found at present is that we do not find the resultant by simply adding the forces together.

Geometric Addition.—To assist us in detecting the solution let us consider a familiar instance in which difference of direction plays a part. Consider three straight roads forming a triangle ABC.

If I walk from A to B, and then from B to C, the result that I have arrived at C is the same as if I had walked from A to C, that is to say, the result, as regards final

position, of adding a movement from B to C on to a movement from A to B is the same as that of a movement from



A to C. We may express this fact by saying that the *geometric sum* of AB and BC is AC.

EXAMPLE.

3. A is due west of B ; and a boat has to sail from A to B. Owing to the direction of the wind it is obliged to sail north-east, then tack and sail south-east. If it runs five miles on each tack how much distance has it made good towards the port?

To completely specify a force we require to know (1) its magnitude, say 5 lbs., (2) its direction,[1] say "vertically downwards," or horizontal and due east.

[1] Some writers regard the direction of a force as ambiguous, and add the word sense, saying that the direction of a force is vertical and its sense downwards. The authors quote and desire to adopt, as expressing their own view, the following remarks of Mr. E. P. Culverwell as to this: "Not only does it seem utterly opposed to common language to say that two persons walking straight away from each other are walking in the same direction but in opposite senses, but it masks the fact that in plane geometry + and − signs indicate changes of 180° in direction; and, further, the word sense seems quite superfluous, as 'same direction' and 'opposite direction' answer the purpose equally well."—*Mechanics*, Preface, p. vii.

It will be noticed that the place at which the force is applied does not form part of the specification of the *force*. The effect of the force in tending to move or break a body on which it acts *will* depend on the place at which it is applied.

Thus a force of 10 lbs. acting vertically downwards is completely specified. The statement "a body is acted on by a force of 10 lbs. vertically downwards" is not complete, for the point in the body at which the force is applied is not stated.

Representation of a Force by a Straight Line.—We can convey the information necessary to specify a force by means of a straight line. From any point draw a straight line in the direction in which the force acts. Make the length of the line proportional to the magnitude of the force.

This line, provided we know the number of pounds which each inch of it represents, and which end is the starting-point, specifies the force.

It is sometimes convenient to insert an arrow-head in the line in order to show the direction of the force.

EXAMPLES.

4. Suppose your paper to represent a horizontal plane. Take a length of an inch to represent a pound, and draw a line to represent a force of 3 lbs. acting towards the north.

5. Draw a line to represent, in a convenient manner, a horizontal force of 250 lbs. due to a west wind.

6. Suppose your paper to represent a vertical plane, and draw a line to represent a forec of 2·4 tons acting vertically downwards.

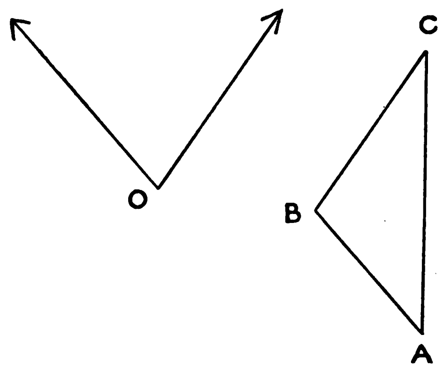
Observe that the representation does *not* include a representation of the point of application of the force.

If the line representing a force is lettered, *e.g.* AB, it is understood that the starting-point is named first. Thus the line AB represents a force acting in the direction from A towards B. An equal and opposite force would be represented by the line BA.

A————————B

Are Forces added together by Geometric Addition?—The fact that a force may be so suitably represented by a straight line may suggest that the method of geometric addition may apply to forces also.

If so the following proposition should be true. If two forces acting at a point O in a plane are represented



by two straight lines AB, BC, their resultant will be represented by the third side, AC, of the triangle ABC, and their equilibrant by the side CA.

By the word "represented" we mean represented on the same scale as regards magnitude, and also represented in direction.

Example.

7. Take the results of your previous experiment (No. 1) and, assuming the proposition just stated, find the resultant of the forces P and Q acting at the given inclination. Compare the result with the value R found for the resultant by experiment.

You will probably conclude that the two values are sufficiently near to justify us in testing the supposition further. What is the discrepancy in the values of the magnitude? What is the difference in the direction of the resultant?

The case in which the lines AB, BC are at right angles is very suitable for a further test, as the necessary calculations are simple, for we know geometrically that in this case

$$AC = \sqrt{AB^2 + BC^2},$$

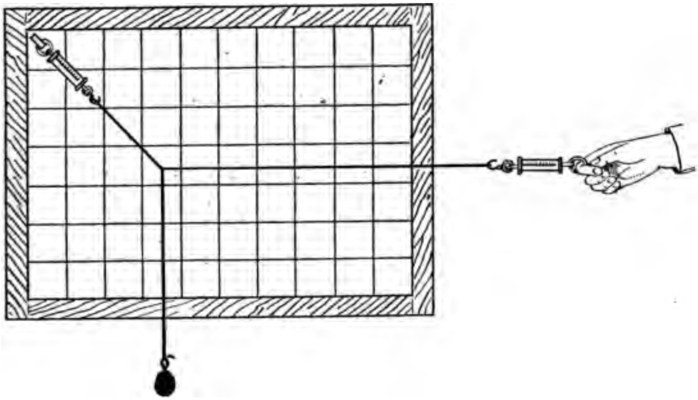
and hence if our supposition is correct we ought to find

$$R = \sqrt{P^2 + Q^2}.$$

Experiment 3.

Fasten a sheet of squared paper to a vertical board, in the top left-hand corner of which a hook is screwed. A spring balance hung from the hook carries a weight of, say, 2 lbs. by a thread. A second spring balance has a thread fastened to it and knotted to the other thread. The second balance is drawn aside by hand, keeping the thread fastened to it horizontal, and the readings of the balances

are taken. The second balance is then drawn aside a little further, still keeping the thread horizontal, and another



set of readings taken. Several readings may be thus taken and the following table completed.

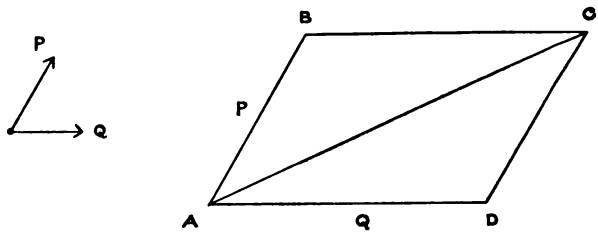
Magnitude of vertical force.	Magnitude of horizontal force.	Magnitude of oblique force by experiment.	Magnitude of oblique force by calculation from the hypothesis.	Discrepancy.
2 lbs.				

Compare the direction predicted by the proposition and found by experiment for the resultant.

Triangle of Forces, Parallelogram of Forces.—We have now some grounds for assuming the truth of the following proposition: *If two forces acting at a point are represented*

by two straight lines AB, BC, *their resultant will be represented in magnitude and direction by the third side,* AC, *of the triangle* ABC.

The question whether the same rule will furnish the resultant of two forces applied at different points of a body, or indeed whether two such forces have a resultant, remains open for the present. Notice that the representation of the forces does not include a representation of their point of application.



We perceive at once that either of the triangles ABC or ADC in the figure would equally serve to represent the two forces P, Q, and their resultant R. But these two triangles make up a parallelogram, lettered ABCD in the figure, and the proposition might therefore be stated thus. If two forces acting at a point are represented by straight lines AB, AD, drawn from a common starting-point A, their resultant is represented by the diagonal AC of the parallelogram whose sides are AB and AD.

Historically the proposition originated in this form, which is universally known as the "parallelogram of forces."

It was first enunciated in a general form by Newton (1642–1727) in the year 1687, and in the same year by the

French mathematicians Varignon (1654–1722) and Lami (1640–1715).

What representation is furnished by the parallelogram of forces of the *equilibrant* of two forces?

It is an obvious consequence of the "parallelogram" of forces that two given forces acting at a point in given directions have only one resultant and only one equilibrant.

EXAMPLES.

8. Assuming the truth of the parallelogram of forces, find the resultant of two equal forces of 5 lbs. acting at angles of 20°, 40°, 60°, 80°, 90°, and 120° with one another. Compare Experiment 2 and examine the effect of each angle being 1° more than it is supposed to be.

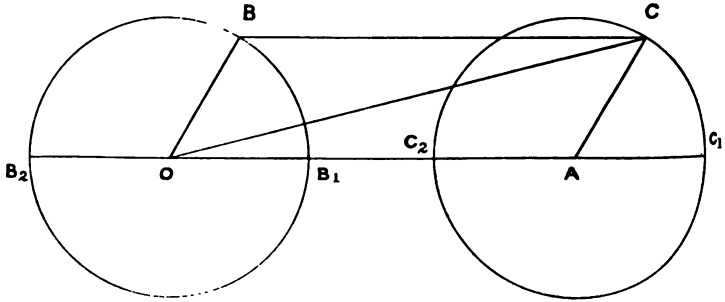
9. Find the direction and magnitude of the resultant of two forces of 8 and 15 lbs. at right angles to one another.

10. The resultant of a force of 10 lbs. and another force inclined to it at 50° is 25 lbs. What is the magnitude of the other force?

It should be noticed that the expression "the angle between two straight lines" is ambiguous, for in general two straight lines make an acute angle with each other and also an obtuse angle. On the other hand, the angle between the directions of two given forces is a perfectly definite angle.

Let us now consider how the resultant of two forces at a point alters as the angle between the forces changes. We may for this purpose suppose that the larger of the forces, represented by OA, remains fixed, while the smaller, represented by OB, revolves about O. The diagonal OC of the parallelogram, of which OA and OB are adjacent sides, represents the resultant. As OB revolves the point B

describes a circle with O as centre, while C describes an equal circle with A as centre. As the angle AOB is diminished the resultant OC approaches the value OA + OB as nearly as we please by taking the angle sufficiently small. So if the angle is increased up to two right angles



the resultant may be made to differ from OA − OB by as small a quantity as we please. For, in the figure it is plain that $OC_1 = OA + OB$ and $OC_2 = OA - OB$. For any intermediate position the magnitude of the resultant will lie between these two values.

EXAMPLE.

11. What is the greatest angle which the resultant of two forces of given magnitudes at a point can make with the greater force? Construct it geometrically. If the larger force is 13 lbs., and the smaller 5 lbs., find the magnitude of the resultant when it makes the greatest possible angle with the larger force. Find this angle.

The experiment No. 1 gave results which were not in *exact* accordance with the proposition. Our previous experience in practical geometry should assist us in detecting possible causes of the discrepancy.

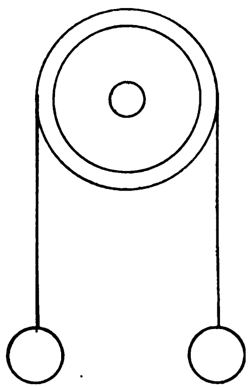
How nearly can you read the spring balances?

How accurately can you fix the positions of the threads?

Find as well as you can the maximum error due to these causes.

If we repeat the experiment with more accurate appliances we shall expect that if the proposition is true we shall obtain experimental results which agree more closely than before with prediction. Instead of spring balances we shall use weights attached to fine threads passing over pulleys.

Use of a Pulley for changing the Direction of a Force.—When a fine thread is placed over a pulley which can turn about a fixed axle or axis so that equal lengths hang down on either side, and equal weights are fastened to the threads, no movement will take place however easily the pulley can turn on its axle. This statement may be regarded as self-evident, or we may compare the pulley to a lever with equal arms (see page 6).



If weights are attached to the threads and the pulley does not move, are we entitled to say that the weights must be equal? If the pulley is perfect, *yes*. But if the pulley is stiff and hard to turn the weights may be far from equal and yet the pulley may not move. Our pulley may be easy to turn and yet not perfect.

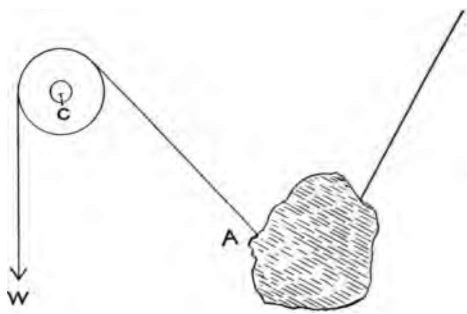
In an actual experiment with equal weights of 0·8

lb. on each side, it was found that an additional weight of 0·009 lb. on one side was required before the system moved. This result indicated that with this particular pulley weights of about this amount hung from the ends of a string passing over the pulley must be equal within, say, 1 per cent., if motion did not ensue.

Experiment 4.

Try the experiment with the pulley supplied to you. Examine the effect of oiling the pulley or adjusting the screws. Try also whether there is any difference in the sensitiveness of the pulley according as the loads are large or small.

If an unknown weight on one side is balanced by a weight of $\frac{1}{2}$ lb. on the other side of the pulley you are using, between what limits does the unknown weight lie?



We frequently use a pulley turning on a fixed axis to enable us to apply a given force to a body by means of a weight as in the diagram.

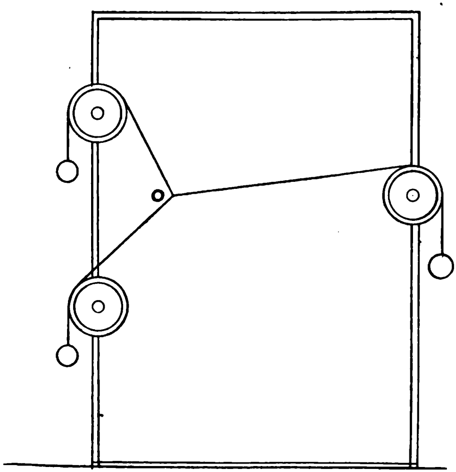
The foregoing experiments will help us to see that the

more freely the pulley runs the more nearly will the pull at A be equal in magnitude to the weight W.

Taking as in previous instances an ideal state of things, to which the reality will only be a more or less close approximation, we say that the tension of a string passing round a smooth pulley at rest is the same on either side.

Experiment 5.

Equilibrium of Three Forces acting at a Point.—Two pulleys are fixed to the edge of a vertical drawing-board and a fine thread is passed over them. To a knot



in the thread between the pulleys attach another thread and fasten weights to the three free ends of the threads. Observe that the knot is acted on by three forces, viz. the tensions or pulls in the three threads. These pulls are by the preceding article equal in magnitude to the three

weights respectively. Any one of these forces is the equilibrant of the other two. If our proposition is correct these three forces are capable of being represented by the sides of a triangle, and consequently any two of them must be greater than the third.

Notice that unless the sum of any two of the weights *is* greater than the third weight the knot will rush in the direction of the largest weight, and there will be a general collapse.

Notice also that if one of the weights is drawn slightly aside and let go, the knot, after swaying to and fro for a short time, should return to its original position. This is a test of the smooth running of the pulleys, and unless it is fulfilled an accurate result must not be expected.

In an actual experiment the middle weight was 0·7 lb. and the side weights 0·5 and 0·4 lb. These are, then, the tensions of the corresponding pieces of thread which keep the knot at rest. The positions of the strings are now marked on a piece of paper fixed to the drawing-board. The lines of the three tensions are then drawn: Ox, Oy and Oz.

Along Ox and Oy measure off lengths OA and OB to represent 0·5 and 0·4 lb. respectively.

Choose a convenient scale, 10 inches to represent 1 lb. may be suitable. Complete the parallelogram OABD. Measure the diagonal OD. Is DOz a straight line?

In the actual experiment described the following results were obtained: magnitude of resultant 0·696 lb. by construction; the difference in direction between DO and Oz was inappreciable.

As an exercise the student should take some unknown weight, such as a piece of iron or lead, and tying it to one of the threads find its magnitude by means of the two known weights which it equilibrates. Then weigh it in a balance in the usual way, and compare the two results.

EXAMPLE.

12. You have a spring balance indicating up to 10 lbs., a few pieces of cord, and a measuring tape. How would you ascertain the weight of a body weighing about 18-20 lbs.?

[This makes a good experiment.]

Could the same method be used for weighing 1 cwt.?

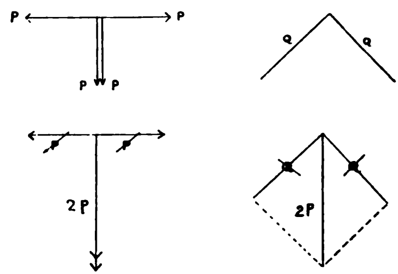
The student must not suppose that a single experiment can establish a universal law. Our conviction of the truth of the "parallelogram of forces" rests on the fact that every carefully-conducted experiment yields results in close agreement with the law.

As in the case of the principle of the lever, which we discussed at p. 13, numerous attempts have been made to deduce the parallelogram law by a process of reasoning from axioms, which every one would accept without dispute.

Thus Daniel Bernoulli (1700–1782) adopts as his axiom that the resultant of two equal forces at right angles must bisect the angle between them, and, further, that the magnitude of the resultant is in this case directly proportional to the magnitude of the original forces.

Consider four forces, each P, acting at a point (see figure, p. 48). The two left-hand forces may be replaced by their resultant, Q suppose, and, by the axiom, the direction of Q bisects the right angle.

The resultant of the forces on the right is also Q. The forces Q are at right angles, and their resultant bisects the right angle. But the resultant of the original forces is



evidently 2P, and therefore the resultant of Q and Q is also 2P. Consequently—

Two forces each P at right angles have a resultant Q,
" " Q " " " " 2P.

Therefore by the second part of the axiom—

$$\frac{P}{Q} = \frac{Q}{2P},$$

whence $Q = P\sqrt{2}$, that is, Q is represented by the diagonal of a square whose side represents P.

By reasoning of this kind Bernoulli finally arrived at the general law.

Such proofs may help to convince a student of the truth of the result, but the parallelogram of forces is now considered as a deduction from experience. For instance, it is experience which convinces us that the magnitude of the resultant of two forces of given magnitude depends

only on the angle between the forces and not on the angle each force makes with the north and south line; or again, that forces in a vertical plane are compounded by the same rule as forces in a horizontal plane.

The parallelogram law may be stated as the law of addition of directed straight lines. Quantities which may be represented in magnitude and direction by straight lines must conform to this law of addition. Other examples of such quantities, which are known as **vector** quantities, will occur subsequently.

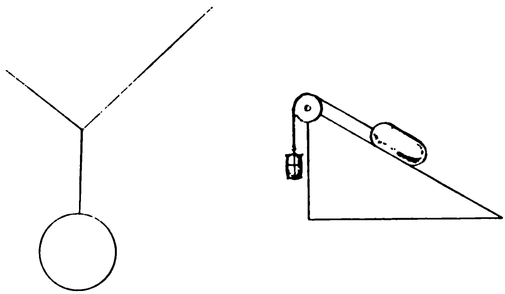
The process by which we have arrived at the parallelogram of forces may seem to be unnecessarily tedious, for the law is not very difficult of comprehension.

That the principle is, however, considerably more subtle than the principle of the lever, may be inferred from the fact that it was not correctly formulated till the time of Newton, about 300 years ago. The principle of the lever, on the other hand, was correctly stated by Aristotle (384–322 B.C.), and was probably known long before his time—*Ὁ οὖν τὸ κινούμενον βάρος πρὸς τὸ κινοῦν, τὸ μῆκος πρὸς τὸ μῆκος ἀντιπέπονθεν.*[1] Or, as we may translate—"The weight that is moved is to the weight that moves in the inverse ratio of the lengths" (of the arms of the lever).

It was not through lack of attempts to discover the solution of this problem that the result remained so long uncertain. In the note-books of Leonardo da Vinci (1431–1519 A.D.), many of which are still extant, the solutions of the problems of the body on the inclined plane and of the

[1] Aristotle, Μηχανικὰ προβλήματα.

body suspended by two strings are attempted over and over again, and the two diagrams sketched here occur with almost endless repetition.



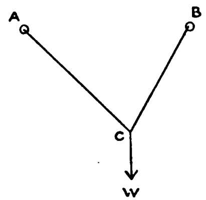
Leonardo himself gives a correct solution of the latter of these two problems, but seems to have been so little satisfied of the accuracy of his result that he gives another completely erroneous solution on the *same page* of his note-book. And this erroneous solution is repeated frequently in subsequent passages.

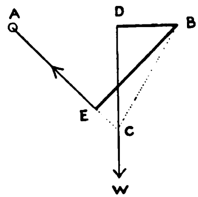
Leonardo da Vinci is universally considered as one of the most versatile of men. There was no branch of literature, science or art in which he did not excel. His paintings in particular, though not numerous, will bear comparison with those of Raphael, and are masterpieces of mediæval art.

Leonardo's correct solution of the problem of the body suspended by two strings is as follows—

If BE and CD are the perpendiculars on AC and AB respectively, we may consider the weight and the pull T in the string AC to act at the ends D and E of the bent

lever EBD, which can move about B as a fulcrum, and





therefore T . BE = W . BD, and in a similar way the tension in BC can be obtained.

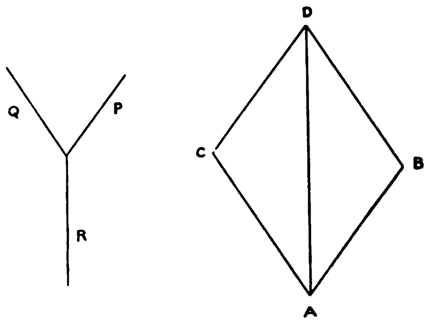
Three Forces in Equilibrium.—The parallelogram of forces enables us to find the resultant or to find the equilibrant of two given forces acting at a point. In practice, however, the problem generally arises in a different way.

Three forces are known to be in equilibrium, that is to say, any one of them is known to be the equilibrant of the other two. We require to find the connection between the forces.

Proposition.—If three forces acting at a point in one plane are in equilibrium, and three straight lines are drawn parallel to the directions of the forces and forming a triangle, then each side of the triangle represents the force to which it is parallel, the scale of the representation being the same for all three sides.

To show that this follows if the parallelogram of forces is true, let P, Q and R be the three forces which are in equilibrium. Take a line AB to represent P, and draw

AC to represent Q on the same scale. Complete the parallelogram ABDC. By the parallelogram of forces AD represents the resultant, and therefore DA represents the equilibrant of P and Q.



But R is the equilibrant of P and Q, and P and Q have only one equilibrant. Therefore DA represents R. That is, the three sides AB, BD and DA of the triangle ABD represent on a certain scale the three forces P, Q and R.

Our knowledge of geometry tells us that any triangle whose sides are parallel to those of ABD is similar to ABD, and hence the sides of any such triangle will equally serve to represent the forces P, Q and R.

We may remind the student that a triangle is completely determined when three independent parts of it are given. Bearing this in mind he should have no difficulty in answering the following questions—

EXAMPLE.

13. Three coplanar forces acting at a point are in equilibrium (coplanar means in one plane). (1) Given their directions, what do you know about their magnitudes? (2) Given, in addition, the magnitude of one of the forces, what do you know about the

magnitudes of the others? (3) Given the directions of the first and second forces and the magnitudes of the second and third show how to find the magnitude of the first and the direction of the third. (4) Given the magnitudes of all three forces, do you know the angles they make with each other? Do you know their directions?

The student should state in each case a corresponding geometrical problem.

In dealing with the following examples the student will find it best to commence by drawing a figure representing the actual physical facts or relative sizes and positions of the objects involved. Then he should reckon up the forces which act, and finally either discover some triangle in the object-diagram which will serve as a force triangle, or, failing this, draw a force triangle.

Examples.

14. A weight of 80 lbs. is suspended by two ropes, 15 and 20 feet long, from two hooks in the same horizontal line 30 feet apart. Find the pull in each rope.

We are not yet in a position to deal with the effect of the weight of the ropes themselves; and in this and subsequent examples we shall put the weight of the ropes out of the question and assume the ropes to be straight. Ropes suitable for the purpose would be of small weight, and consequently we shall obtain results of real practical value, notwithstanding our assumption. We commence by drawing the diagram of the object to scale. The student will have no difficulty in this—for he has simply to draw a triangle whose sides are in the proportion of 30 : 20 : 15. He should indicate the direction of the weight, which must be at right angles to AB.

We next observe that the given weight, 80 lbs., is the equilibrant of the pulls, say T_1 and T_2, in AC and CB.

Draw a line, KL, parallel to the direction of the weight, to represent 80 lbs. A convenient length for this purpose may be 4 inches. Through K and L draw parallels to AC and CB, intersecting in M. We thus obtain one of the two triangles shown

of 3 feet from both CA and CB. A weight of 14 lbs. is ...led at D by two strings, DA and DB, attached to A and B. ...e tension of each string.

A light string, which will not stretch, of length 4 feet 2 inches ...ended from one end, and it is found that the greatest weight ...n be attached to the other end without the string breaking is ...2 ozs. If the same string had been attached to two points in ...me horizontal line 4 feet apart, what would have been the ...st weight which could have been attached to the middle point ...ut the string breaking?

A case weighing a ton is slung by a rope 30 feet long from a support vertically overhead. The case is to be drawn aside a ...ontal distance of 10 feet by a horizontal force and held in that ...ion. What horizontal force is required? Through what ...ical height is the weight raised? What is the tension of the ...porting rope?

...9. A weight of 15 lbs. is supported in a vertical plane by two ...es of 10 and 12 lbs. Find the angles which those forces will ...ke with the vertical.

20. A weight of 15 lbs. is supported by two strings, which make ...gles of 120° and 140° with the vertical. Find the tensions of the ...o strings.

21. A weight of 65 lbs. is supported ... pieces of the same ...ire equally inclined to the vertica... breaking tension of ...e wire is 45 lbs., what is the great... at the two parts of ...ire can make with one ...

22. Three forces of 2... the same ...oint in directions inc... Find ...e resultant graphi... nd the ...ngle it makes wit...

23. What hori... force of ...50 lbs. acting ...

24. Find by ... 100 lbs. ...ting at an ...

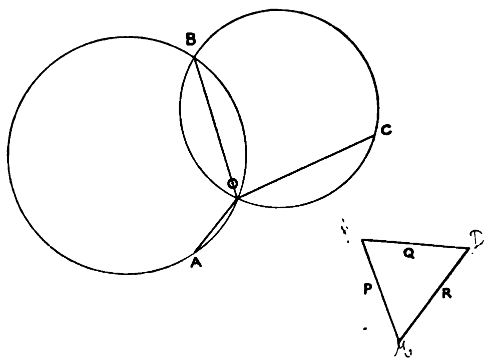
25. A ... ch of two ... he pulleys ... rted 5 feet ... Neglecting ... find (i) the ... crease if the

weight is to be held 15 feet from the ground; (iii) the horizontal pull on each mast in each case, supposing the rope to come down vertically from each pulley to the hauling end.

At what angle should the rope be led from the pulley if there is to be no horizontal pull on the mast?

26. Two pegs, A and B, support a cord which measures 10 feet between the pegs, and carries a weight of 20 lbs. by means of a second cord knotted to it at a point 4 feet from the end attached to the peg A. Draw the cord in position and find the tension in each of the two parts into which the knot divides it. B is 1 foot above A and 6 feet to the right of A. [Army]

27. Three strings are knotted together at O, each string passes over a small pulley, which may be treated as a point, and carries a given weight. Given the position of the three pulleys A, B, C, construct the position of the knot.



Let P, Q, R be the three weights for the pulleys at A, B and C respectively.

Draw a triangle, DEF, whose sides are proportional to P, Q and R, so that EF represents P, FD represents Q, and DE represents R.

On AB construct a circle, such that the angle on AB is equal to the angle F.

[There are two such circles. Will either one do? If not, show how to select the right one.]

On BC construct a circle, such that the angle on BC is equal to the angle D.

The point of intersection other than B of these circles is the position of O.

Take BC vertical: AB = 30 inches, AC = 28 inches, BC = 26 inches, P = 7 lbs., Q = 8 lbs., R = 9 lbs.

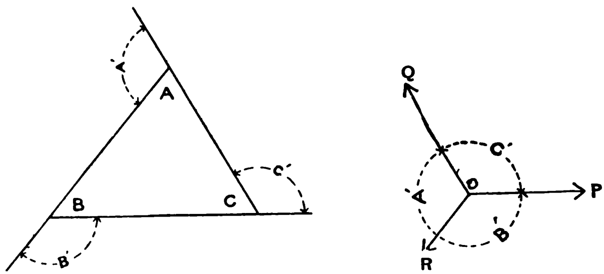
Make the construction, and measure the distance of O from A, B and C.

[Many additional examples will be found in the following chapter.]

CHAPTER III

Analytic Methods.—It is frequently more convenient to find the relation between three forces acting at a point, which are in equilibrium, by the aid of trigonometry instead of by drawing a diagram to scale. We proceed now to investigate the methods for dealing with such cases.

The triangle ABC, which has its sides parallel to the



three forces in equilibrium, P, Q and R, acting at O, has also its sides proportional to these forces. We may, therefore, write down the equations $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$, when a, b and c are the sides of the triangle ABC. But we know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, and therefore it follows that—

$$\frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}$$

If we produce the sides of the triangle ABC and call the exterior angles A′, B′ and C′ it is at once apparent that A′ is equal to the angle which the forces Q and R make with one another; similarly, B′ is equal to the angle opposite Q, and C′ is equal to the angle opposite R. Therefore, since $A' + A = B + B' = C + C' = 180°$, we have—

$$\sin A = \sin A',$$
$$\sin B = \sin B',$$
$$\sin C = \sin C',$$

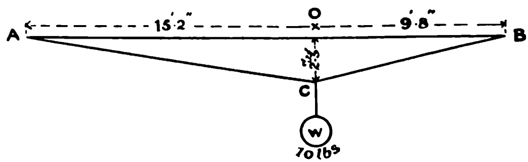
and consequently $\frac{P}{\sin A'} = \frac{Q}{\sin B'} = \frac{R}{\sin C'}$, or in words: If *three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.* This result is generally known as *Lami's theorem*, as it was first enunciated by Father Bernard Lami (*Traité de Méchanique*, 1687).

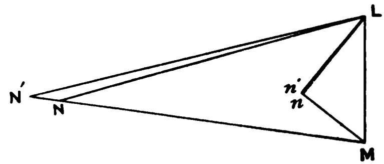
Example.

1. A piece of strong fishing line is fixed to two nails A and B in the same horizontal line, and a weight of 10 lbs. is suspended from a point C on the line. AB is measured to be 25 inches, and the perpendicular distance of C from AB, viz. CO = 2·3 inches and AO = 15·2 inches. Find the tensions in the two parts of the line.

This example is best done by a method of calculation: if we attempt to draw a triangle of forces, since the inclination of the two parts of the string AC and CB is not far off 180°, a small error in drawing our parallel lines will produce a considerable error in the resulting tensions. This will be clear from the diagram (p. 60), where LM represents the weight of 10 lbs. and MN and LN are parallel

to the two parts of the string. Let us suppose that the angle M is drawn accurately, but the angle L is made too small by 2°.[1] The corresponding error in the length of MN is NN′: this error is about $\frac{1}{13}$ of the whole length to be measured. On the other hand, if the two parts of the string had





been inclined at an angle of about 90° (say), the error produced in the length of MN by an error of 2° in the inclination of LN would have been a much smaller percentage of the whole. In the figure nn' is only $\frac{1}{35}$ of Mn.

Let T_1 be the tension in CA and T_2 the tension in CB,

$$\tan ACO = \frac{15{\cdot}2}{2{\cdot}3} = 6{\cdot}62,$$

$$\therefore ACO = 81^\circ\ 20',$$

$$\text{so } \tan OCB = \frac{9{\cdot}8}{2{\cdot}3} = 4{\cdot}26,$$

$$\therefore OCB = 76^\circ\ 45',$$

$$\therefore \angle\ ACB = 158^\circ\ 5';$$

[1] The error of 2° is much larger than would be made with careful *drawing; it is taken*, however, for clearness in the figure.

and by Lami's theorem—

$$\frac{10}{\sin \mathrm{ACB}} = \frac{\mathrm{T}_1}{\sin \mathrm{OCB}} = \frac{\mathrm{T}_2}{\sin \mathrm{ACO}},$$

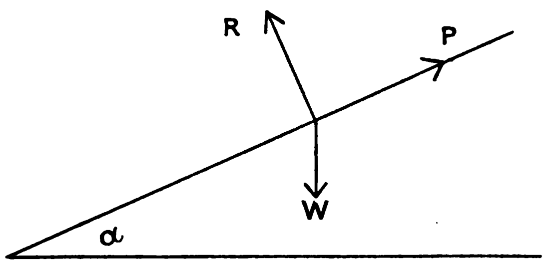
$$\frac{10}{\sin 158^\circ\ 5'} = \frac{\mathrm{T}_1}{\sin 76^\circ\ 45'} = \frac{\mathrm{T}_2}{\sin 81^\circ\ 20'},$$

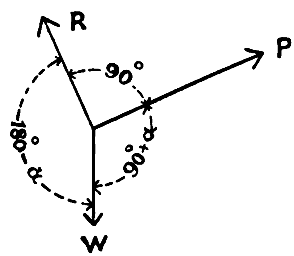
$$\frac{10}{0{\cdot}374} = \frac{\mathrm{T}_1}{0{\cdot}973} = \frac{\mathrm{T}_2}{0{\cdot}989},$$

giving $\mathrm{T}_1 = 26{\cdot}0$ lbs., $\mathrm{T}_2 = 26{\cdot}4$ lbs.

Examples on Lami's Theorem.

2. We may define a *smooth* plane as one which can only exert a thrust at right angles to its surface.





A weight of W lbs. is supported on a smooth plane inclined to the

horizontal at $a°$ by a force parallel to the lines of greatest slope. Find the force necessary, and the pressure on the plane.

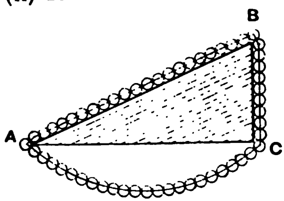
The forces acting are shown in the diagram. By Lami's theorem—

$$\frac{P}{\sin(180 - a)} = \frac{R}{\sin(90 + a)} = \frac{W}{\sin 90},$$
$$i.e.\ P = W \sin a,$$
$$R = W \cos a.$$

3. If in the preceding question P is horizontal, find R and P in terms of W and a.

4. What weight will a horizontal force of 20 lbs. support on a smooth plane whose inclination is 27°?

5. Stevinus (1548–1620) solved[1] the problem of a weight resting on an inclined plane by arguing that (i) An endless uniform chain placed round the plane would be at rest. Do you admit this? (ii) It would still be at rest if the festoon or part hanging freely were removed. (iii) The pull at B must be equal to the weight of the part of the chain which is vertical. (iv) Therefore the pull at B necessary to support the length AB on the plane is equal to the weight of BC. Find whether this solution gives the same result as that given by Lami's theorem.



6. A weight of 10 cwts. suspended by a chain 30 feet long is drawn aside and held by a horizontal rope whose tension is 1 cwt. What is the tension of the chain, and through what vertical distance has the weight been displaced?

7. Find the force necessary to support a weight of W lbs. on a smooth inclined plane whose inclination is $a°$ if the force is inclined to the horizontal at $\theta°$.

Give the numerical result if

$$W = 100 \text{ lbs.},$$
$$a = 15°,$$
$$\theta = 41°.$$

8. A weight of 500 lbs. is hung up by a rope 20 feet long. What

[1] Stevinus was so pleased with his solution that he adopted the diagram, with the proud motto, "The wonder is a wonder no longer," as the title-page of his book.—Mach, *Mechanics*, p. 30.

horizontal force is required to hold it at a distance of 3 feet out of the plumb? By how much is the tension of the rope increased?

9. A piece of wire, 32 inches long, whose breaking tension is 85 lbs., is fixed to two points in the same horizontal line 30 inches apart. Find the load which, suspended at the centre, may be expected to break the wire.

10. A weight of 15 lbs. is supported by a horizontal pull, and a string inclined at $a°$ to the vertical. Calculate the horizontal pull and the tension of the string. (The results may be compared with those found by experiment, p. 39.)

11. A string passes round a nail. The two portions of string being inclined at an angle a, and the tension of the string being 50 lbs., find the resultant pull on the nail. Tabulate the results when—

$$a = 20°, 40°, 60°, 80°, 100°, 120°, 140°, 160°.$$

12. A weight of 160 lbs. is slung by two cords inclined to the horizontal at 40° and 70° respectively. Find the tensions in the cords.

13. Find the magnitude of two equal forces acting at an inclination of 174°, whose resultant is 180 lbs.

14. A cord fastened to a point A passes through a ring, from which a weight W can be hung, then over a pulley B at the same level as A, and has a weight P hanging from it. Calculate the weight W necessary to make the ring rest a given depth below AB.

Application: Let P = 10 lbs.,
AB = 20 inches.

Depth of ring, 5, 10, 15, 20, 25 inches below AB.

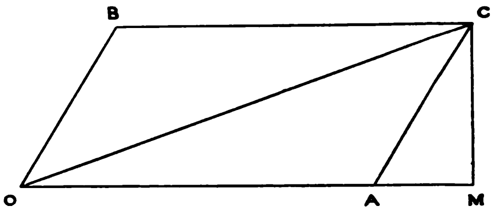
Calculate the corresponding values of W. (Assume the tension of the cord is P throughout.)

Resultant by Calculation.—Given two forces, P, Q, acting at a point and the angle a between their directions, to calculate the magnitude and direction of their resultant—

Let OA and OB represent P and Q. Thus $\angle$ BOA = a. Completing the parallelogram OACB we know that OC represents the resultant.

Draw CM perpendicular to OA.

The figure is drawn for the case of a being an acute angle.



In the triangle CAM—

$$\angle \mathrm{CAM} = a,$$
$$\mathrm{CM} = \mathrm{AC} \sin a,$$
$$\mathrm{AM} = \mathrm{AC} \cos a,$$
$$\therefore \mathrm{OM} = \mathrm{OA} + \mathrm{AC} \cos a.$$

In the triangle OCM—

$$\begin{aligned} \mathrm{OC}^2 &= \mathrm{OM}^2 + \mathrm{CM}^2, \\ &= (\mathrm{OA} + \mathrm{AC} \cos a)^2 + (\mathrm{AC} \sin a)^2, \\ &= \mathrm{OA}^2 + \mathrm{AC}^2 + 2\mathrm{OA} . \mathrm{AC} \cos a, \\ &= \mathrm{OA}^2 + \mathrm{OB}^2 + 2\mathrm{OA} . \mathrm{OB} \cos a. \end{aligned}$$

Now the magnitudes of (*e. g.* number of pounds in) P, Q and R are proportional to the lengths of (*e. g.* the number of inches in) OA, OB and OC respectively. Consequently—

$$\mathrm{R}^2 = \mathrm{P}^2 + \mathrm{Q}^2 + 2\mathrm{PQ} \cos a.$$

Draw a figure for the case when a is an obtuse angle, and verify that the same formula holds good.

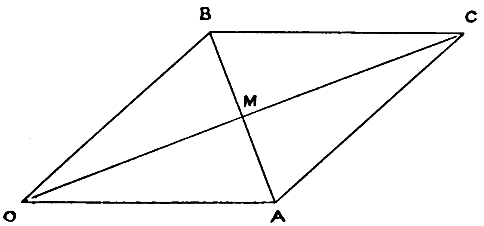
Remark.—Students are occasionally confused as to the sign + or − of the term 2PQ cos α. To avoid this, notice that the smaller α is the larger R is, so the sign must be +.

To find the direction of R let $\angle$ AOC = θ. Then in the triangle COM we have—

$$\tan\theta = \frac{CM}{OM} = \frac{OB\sin\alpha}{OA + OB\cos\alpha},$$

$$= \frac{Q\sin\alpha}{P + Q\cos\alpha}.$$

Two particular cases, from their frequent occurrence, are worthy of notice.



I. **Equal Forces.**—P = Q. Then the parallelogram OACB becomes a rhombus whose diagonals are therefore at right angles. Let the diagonals intersect in M.

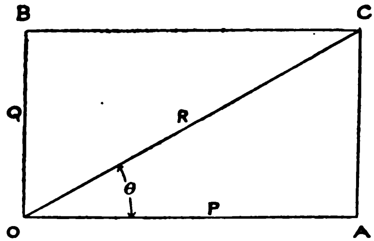
Then $$\angle AOM = \frac{\alpha}{2},$$

$$OM = OA\cos\frac{\alpha}{2};$$

$$\therefore OC = 2OA\cos\frac{\alpha}{2},$$

and $$R = 2P \cos \frac{a}{2},$$

and obviously $$\theta = \frac{a}{2}.$$



II. **Forces at Right Angles.**—$a = 90°$. The triangle COA is a right-angled triangle, and

$$OC^2 = OA^2 + OB^2,$$

whence $$R^2 = P^2 + Q^2.$$

Further, $$\tan \theta = \frac{AC}{OA} = \frac{Q}{P}.$$

EXAMPLES.

15. Two men pull a heavy box by means of ropes, the tension in each rope being 160 lbs. If the inclination of the ropes is 30°, what is the force tending to move the box?

16. Find the resultant of forces of 12 and 15 lbs. inclined at an angle of 60°.

17. Two forces of 11 and 17 lbs. have a resultant of 8 lbs. What is the angle between them?

18. What angle does the resultant of forces of 1 and 10 lbs. inclined at 70° make with the larger force?

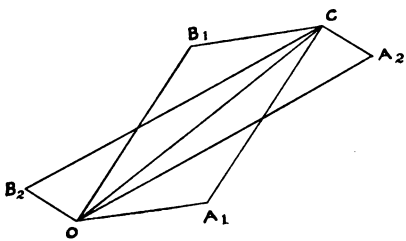
19. Find the resultant pull due to two pulls of 100 and 120 lbs. at an inclination to one another of 10°.

20. Find the resultant force on a roof due to (1) a vertical load of

10 lbs. per square foot, (2) a wind pressure at right angles to the roof of 20 lbs. per square foot. Area of roof surface 200 square feet. Inclination of plane of roof to horizontal 25°.

The forces P and Q, which have R as their resultant, are called the components of R. The problem—to draw a parallelogram, given one diagonal, has an infinite number of solutions. Thus a force may be separated into two components in an infinite number of ways. But there is only one way of separating a force into two components in *given different* directions.

For instance (see figure), the force represented by OC may be separated into components represented by OA_1, OB_1, in directions making given angles say 30° and 20° with OC. Similarly, OA_2 and OB_2 represent components in directions making 10° and 110° with OC.



We may separate a force represented by OC into components acting in any two straight lines through the point of application of the force. But the directions, from or towards the point, of these components are not at our disposal. A force acting at a point on a rigid body may be replaced by its components without affecting the equilibrium of the body (see p. 73).

EXAMPLE.

21. Find the components of R in directions making angles of 90° and $x°$ respectively with that of R.

The case of the components being at right angles is by far the most important, because a force has no tendency to move a body in a direction at right angles to its own direction. Thus, take the case of a body which can only move in one line, such as a railway truck on the level or the drawer of a table. A force parallel to the handles could never pull the drawer out, and a force at right angles to the rails could never move the truck along the rails.

To find to what extent an oblique force tends to move the truck along the rails, let us separate it into two components, one parallel and the other perpendicular to the rails. The component perpendicular to the rails has no tendency to move the truck along the rails, and the useful effect, in moving the truck along the rails, of the original force is the same as that of the component parallel to the rails.

EXAMPLE.

22. In towing a boat along a river, is it easier to tow with a long rope or a short one, and why? (The weight of the rope is to be neglected.) [Army]

Resolving.—It is convenient to have a name for the process of breaking a force into two components at right angles, and we shall use the word "resolving" for this purpose. We shall speak of these components as the *resolved parts* of the force in the given directions. We

have already noticed that we cannot attach any definite meaning to the term "the component of a force in a given direction." The *resolved part* of a force in a given direction is, however, perfectly definite.

The resolved part of a force R in a direction making an angle θ with that of R is R $\cos \theta$.

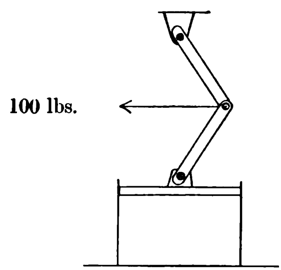
Can a force be replaced by its resolved parts in any two given directions?

EXAMPLES.

23. Two equal forces acting at 175° have a resultant of 100 lbs. Find the magnitude of the forces.

The property indicated by this example is turned to account in the Stanhope levers, so termed because the invention was applied by Lord Stanhope to a printing press in 1798.

Very great forces in the hinged bars are requisite to balance the pull of 100 lbs. The result is that the plate hinged to the lower lever is pressed with great force against the bed-plate.



24. Three forces act at a point as shown in the figure on the next page. Find their resultant.

In this case it will be best to replace the forces in OB and OC by their components along OA and at right angles to it.

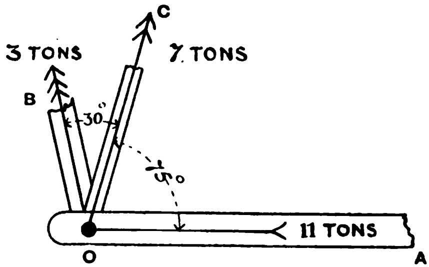
We thus see that the original three forces have the same resultant as forces x along OA, and y, upwards perpendicular to OA.

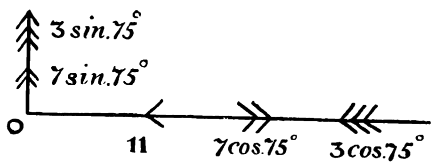
where $$x = -11 + 7 \cos 75^\circ - 3 \cos 75^{c},$$

and $$y = 7 \sin 75^\circ + 3 \sin 75^\circ,$$

$$x = -11 + 4 \cos 75°,$$
$$= -11 + 1·07,$$
$$= -9·93,$$
$$y = 10 \sin 75° = 9·65.$$

x being a negative force along OA really acts along AO.





The resultant R acts at an angle θ with AO *produced*

where $$\tan \theta = \frac{9·65}{9·93}$$

so that $$\theta = 44° \ 14'$$

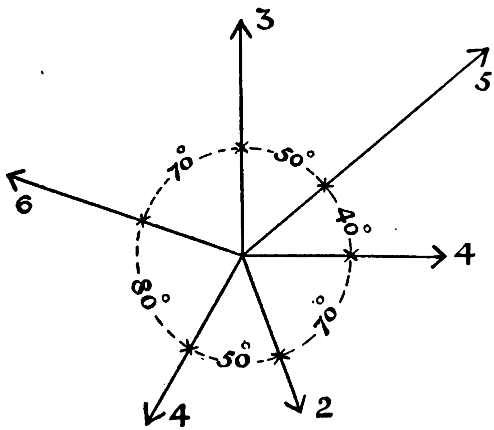
and $$R = \frac{9·93}{\cos \theta} = 13·85 \text{ tons.}$$

Check by constructing the resultant graphically, re*presenting* the forces 11, 7, 3 by red lines, x y by blue *lines and R by* a black line.

Examples.

25. Find the resultant of four equal forces, each 70 lbs., the line of action of each making an angle of 10° with that of the next.

26. Find the resultant of the set of forces shown in the diagram.



27. What will be the resultant of five forces, of 1, 2, 3, 4 and 5 lbs., directed from the centre towards the angular points of a regular pentagon?

[It may be shown that

$$\cos 72° = \sin 18° = \frac{\sqrt{5}-1}{4},$$

$$\cos 36° = \sin 54° = \frac{\sqrt{5}+1}{4}.$$

Using these results, x and y may be expressed numerically without using tables.]

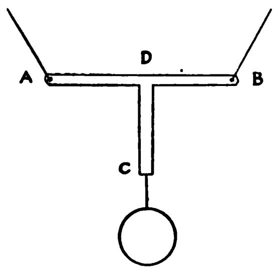
28. Three forces, P_1, P_2, P_3, act at a point O and make angles a_1, a_2, a_3 with a fixed line Ox. Find the magnitude of their resultant and the angle its direction makes with Ox.

If $P_1 = 20$ lbs., $P_2 = 30$ lbs., $P_3 = 40$ lbs., and if the lines of action of P_1 and P_3 make angles of 45° and 60° with that of P_2, but on opposite sides of it, find, graphically or otherwise, the magnitude of the resultant to the nearest pound.

Rigidity.—We have up to the present supposed that the three forces acting on a body and keeping it in equilibrium are applied at one and the same point. This will often not be the case in actual practice, and we now consider the condition that three forces applied in one plane to a body may keep it in equilibrium.

Some introductory remarks are necessary.

Common-sense tells us that forces applied at different points of a body may break or bend it, although equal forces applied at one and the same point would keep it in equilibrium.



Consider, for instance, the figure which represents a small T-shaped piece of wood, hung up by strings from A and B, and carrying a weight at C. It is obviously likely to break at D, whereas had equal forces been applied at D it would very likely have been safe.

We realize by similar instances that a set of forces acting on a body may (*a*) distort it, that is, alter its shape, and perhaps break it, or (*b*) may move it bodily, or (*c*) may, to all appearance, leave it undistorted and unmoved.

A body which does not undergo any distortion in consequence of the application of force to it is called a *rigid body.*

An ideal rigid body would be one which would not *suffer any distortion* at all, either of shape or size, under

the influence of any forces however great. Even the strongest known substances, such as steel, are not perfectly rigid, though the want of rigidity, under the action of small forces, can only be detected by the use of delicate appliances.

Many bodies are, however, to all ordinary intents and purposes rigid when subjected to only moderate forces; that is to say, their distortion is inappreciable by ordinary means, and we shall assume that we are dealing with such bodies.

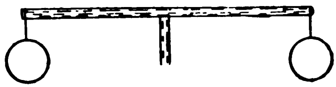
Any forces acting *on a rigid body* may be replaced by their resultant without affecting the equilibrium of the body. It is this proposition which justifies us in regarding the weight of a rigid body as a single force, and in treating the pressure of a book on a table as a force acting at a single point.

A force can never in reality act *at* or *on a point*. For consider a needle pressed against a table with a force of 1 lb. If the area over which the pressure is distributed were only $\frac{1}{200000}$ square inch, we should have a pressure at the rate of 200,000 lbs. to the square inch, which not even a steel plate could withstand.

Another proposition obviously true of rigid bodies, and obviously not to be assumed of bodies whose rigidity under given forces is to be examined, is this: A force may be supposed to be applied at any point in its line of action.

If we are endeavouring to decide whether a particular body will be rigid when acted on by given forces, we *must not assume* that two forces may be replaced by

their resultant, or that the weight of the body acts at its centre of gravity.



For instance, if a plank rests horizontally across a post and carries 1 cwt. at each end, and we want to know whether the plank will break, it will be ridiculous to replace the two weights by their resultant.

Proposition.—*If three forces acting in one plane on a rigid body keep it in equilibrium, their lines of action must either be parallel or meet in a point.*

We have seen in Chapter I. that their lines of action may be parallel. Suppose, if possible, that they are not parallel and yet do not meet in a point.

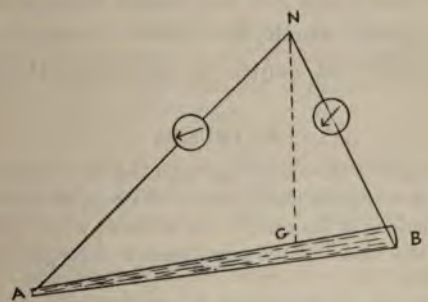
Then, the lines of action of two of them must meet, say, at O. Imagine a smooth pin put in at O. This cannot *destroy* equilibrium. But the third force, which by supposition does not act through O, will cause the body to turn round O, and there will be no force to prevent this turning. Thus equilibrium will not subsist even when the pin at O is inserted, and *a fortiori* it cannot subsist when the pin is taken out.

Therefore equilibrium is impossible unless the direction of the third force also passes through O.

Experiment 1.

Take a bar of variable section like a billiard-cue or a poker. Find its centre of gravity by balancing on a knife-

edge and mark the point, also determine its weight. By means of small staples or hooks at its ends hang it up by two cords to a nail N on a vertical blackboard (the nail ought to be fairly long so as to keep the bar from touching the board). The pulls in the two cords may be determined by spring balances. How could you test whether the centre of gravity lies vertically below the nail? Obviously by hanging a weight by a thread from the nail so as just to clear the rod and noticing whether the thread passes exactly in front of the mark on the rod. You will



probably conclude that in the position of equilibrium the centre of gravity of the rod is vertically below the nail.

What are the forces acting on the rod? Do they satisfy the condition stated on page 74?

Draw a vertical line on the blackboard or on a sheet of paper pinned to it, and also lines parallel to the strings. Draw a vertical line to represent the weight and through its ends draw parallels to the strings.

Are the three sides of the triangle thus drawn

proportional in length to the weights and the pulls in the cords?

You will probably conclude that the result justifies us in assuming the truth of the proposition on page 74 that the three forces which keep the rod in equilibrium may be supposed to act at N.

The experiment may be varied by attaching the cords to different points, so that the parts of the strings make (1) acute angles and (2) obtuse angles with the rod. Observe that in all cases the two strings lie in the same vertical plane (how would you test this?), and when produced they meet in a point which lies either vertically above or vertically below the centre of gravity of the rod.

EXAMPLES.

29. If the ends of the beam were attached to a single cord which was slung over a smooth pulley, what inference, as to the inclination of the two portions of the cord to the vertical, would you draw from the fact that the cord has the same tension throughout? Test this by experiment.

30. If the cords were parallel how could you determine their tensions?

31. A picture weighing 25 lbs. is hung up by a cord 48 inches long passing over a nail, and tied to points in the frame 18 inches apart. Find the tension in the cord.

32. A string with one end fixed runs horizontally to a pulley, passes round the pulley, and carries a weight of 56 lbs. at the other end. Find the thrust of the pulley on its axle, neglecting the weight of the pulley and the cord. Give its magnitude to two significant figures and state its direction. [Army]

33. If the picture-cord used cannot safely bear a tension of more than 50 lbs., what is the shortest length of cord suitable for hanging up the picture in Example 30?

34. The rafter of a roof is acted on by (1) the load it bears, which may be supposed to act at the middle point; (2) a horizontal thrust

by the other rafter to which it is joined at the top ; (3) the thrust of the wall on which it rests. The load being 200 lbs. weight and the rafter inclined at 28°, find the thrust of the wall. Write down its magnitude and its inclination to the horizontal.

35. **The Bent Lever.**—ABC is a bent lever whose fulcrum is at B. Forces P and Q act at A and C at right angles to AB and BC.

If $AB = a$, $BC = b$, $\angle ABC = \alpha$, find the condition of equilibrium and the pressure on the fulcrum.

36. Three equal strings, AB, BC, CD, each 1 foot long, are knotted together, and suspended from two points A, D in the same horizontal line and 2 feet apart. Weights of 2 lbs. are hung from B and C.

Draw the position of equilibrium and draw triangles of forces for the forces acting on the knot at B and the knot at C. Hence find the tensions of the three strings. Compare the results with those furnished by calculation.

[An arrangement of this kind has been suggested as a balance. If BC is horizontal the weights at B and C must be equal. Try this experimentally. Examine the accuracy of the machine. Point out as many sources of error as you can.]

37. A uniform heavy rod rests, in a horizontal position, with one end on a smooth plane (inclination α° to horizontal) and the other end supported by a string. Find the direction and tension of the string and the pressure on the plane, if the rod weighs $2\frac{1}{2}$ lbs., and is 18 inches long.

[Notice that the direction of the string is determined by the fact that the lines of action of the three forces on the rod must meet in a point.]

38. A rail ABC, 10 feet long, in a horizontal position, is bent into a circular arc of 20 feet radius by a force of 1 ton applied at its middle point B. The rail rests against two smooth supports at its ends A, C. What are the pressures against these supports?

39. A solid ball weighing 40 lbs. and measuring 6 inches in diameter is hung by a string of length 2 inches fastened to its surface, and to a smooth wall against which the ball rests. Show that the tension of the string is 50 lbs.

Many of the following examples may be dealt with either graphically or by calculation. Except where specially indicated the choice of method is left to the student.

40. Two weights are knotted to a string whose weight may be neglected, and hung up by means of the string. The part of the string between the weights is inclined at 18° to the horizontal, while the other two parts are inclined at 46° to the horizontal. Compare the two weights. [Hint for solution: Consider the equilibrium of each knot under the three forces acting there.]

41. A uniform rod rests over a smooth peg with its end against a smooth vertical wall 6 inches from the peg, and is inclined at 55° to the vertical. Find the length of the rod.

42. The two ends of a string passing over a smooth pulley are tied to the same body. Show that in the position of equilibrium the centre of gravity of the body is vertically below the pulley and that the two portions of the string make equal angles with the vertical.

43. A uniform bar 3 feet long, weighing 17 lbs., can turn freely in a vertical plane about its upper end. It is held at an angle of 60° to the horizon by two forces, a horizontal one of 2 lbs. applied at the lower end and one of 8 lbs. at a point 1 foot 9 inches from the upper end. Both forces lie in the vertical plane containing the bar and act towards the same side. What angle will the second force make with the bar in the position of equilibrium?

44. Six bags of coal, being hoisted from the hold of a collier, are hanging vertically by a rope 30 feet long. Find the horizontal force necessary to pull them 15 feet from the vertical. [Ten bags go to the ton.]

45. A river is 48 yards in width. Two men walking abreast on opposite sides, at a distance of 1 yard from the bank, draw a boat with ropes 30 and 40 yards long, pulling with forces 200 and 150 lbs. respectively. Show that the resulting force is in the direction of the river, and find its magnitude.

46. A picture is hung by a string passing over a nail and tied through two rings on the picture. Show that the shorter the string the greater is the strain on it.

47. A uniform rod AB is suspended from a nail C by two strings CA and CB. Show that the tensions of the strings are proportional to their lengths.

48. A derrick consists of a spar 28 feet long, the lower end at the foot of a mast, the upper end carrying a pulley and supported by a chain 17 feet long fastened to the mast at a height of 20 feet. A weight of 800 lbs. is raised by a chain passing over the pulley at the end

of the spar and pulled parallel to the spar. Find the thrust on the spar and the pull in the chain that supports it.

49. A cord ABCD is fastened at A and D, and at B is hung a weight of 10 lbs. and at C an unknown weight of W lbs. The inclinations of AB, BC and CD to the vertical are 35°, 72° and 56° respectively. Find the unknown weight.

50. A body is suspended by two cords from fixed points, and the tensions of the cords, as measured by spring balances inserted in their lengths, are 4·3 and 6·7 lbs. The inclination of the cords to one another is 50°. Find graphically or otherwise the weight of the body.

51. Two forces of 4 and 6 lbs. are to be employed to hold a body weighing 7 lbs. in a vertical plane. Find the angles which these forces will make with the vertical. [A graphical solution, drawn to scale, will suffice.]

52. A rod AB, whose length is 6 feet 8 inches, and whose weight is 1 lb., is hinged at B, and is tied by a light string AC 5 feet long to a peg C such that BC is horizontal and of length 8 feet 4 inches. If the rod is in equilibrium, find the tension of the string, and the reaction at the hinge.

53. A horizontal force of 4 lbs. weight just keeps a weight of 9 lbs. at rest on a smooth inclined plane. Find the inclination of the plane to the horizon.

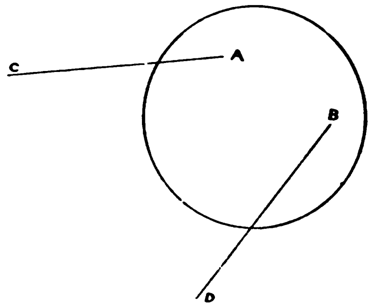
54. A thin steel wire is fastened to two points in the same horizontal level 6 feet apart. A weight of 3 lbs. is hung by another piece of wire which is fastened to the centre of the 6-foot length of wire, and which causes a deflection of 1 inch in the centre of the horizontal wire. Determine in any way you please the pull on the end points to which the 6-foot wire is attached. [Army]

55. The least direct pull which will break a given wire is 85 lbs. A piece of this wire 3 feet long is fastened at its ends to points in the same horizontal line 2 feet 10 inches apart. Find the smallest weight suspended at the middle point which will break the wire.

56. A weight of 100 lbs. is supported by two ropes inclined at 27° to the horizontal. Find the pull on each rope.

57. A circular disc lying on a smooth table has strings attached to it at A and B which are pulled with forces represented by the lines AC and BD (an inch representing 1 lb.). A string is to be attached and pulled so as to keep the slab at rest. Determine a point E at

which it may be attached and find the magnitude and direction of the pull. [Army]



58. A weight of 600 lbs. is suspended from a hanging pulley by a rope which, passing over the pulley, leaves it at an angle of 60° to the vertical in a downward direction. Find approximately the tension of the rope suspending the pulley and the direction in which it rests.

59. A weight of 28 lbs. hangs by a light ring to which are attached strings which pass over fixed pulleys and carry weights of 12 and 20 lbs. respectively, hanging vertically. Find the angle between the strings which meet at the ring. What happens if an additional weight of 4 lbs. is attached to the ring?

60. A weight C of 500 lbs. is suspended by two ropes AC, BC, lengths 5 and 11 feet respectively, from two hooks A and B, which are in the same horizontal line and 12 feet apart. Find approximately, either graphically or by calculation, the tensions of the ropes. Determine also the greatest horizontal force which can be applied at C parallel to AB without disturbing the equilibrium.

61. A vertical force of 120 lbs., a horizontal force, and a force of 240 lbs. keep a particle in equilibrium. Find the horizontal force, and the direction of the inclined force.

62. A heavy uniform rod weighing 70 lbs. hangs from a hinge, and is held at an inclination of 20° to the vertical by a force applied at right angles to the rod at its lower end. Find the magnitude of this force and the reaction of the hinge.

63. A toothed wheel can roll in a vertical plane on another fixed wheel, the teeth of both being small. If the movable wheel is supported by a horizontal force through its highest point in a

osition when the line of centres makes 30° with vertical, find the atio between the horizontal force and the weight of the wheel, which is supposed to act through its centre.

64. A plank is placed in a horizontal position, with its ends on two smooth planes, which are inclined to the horizontal at 55° and 35° respectively. Neglecting the weight of the plank, find at what point a weight must be placed in order that equilibrium may be possible.

65. A circular board, 2 feet in diameter, has three smooth pulleys fixed at points 120° apart on the rim. Strings passing over the pulleys are attached to a small ring, and carry weights of 2, 3 and 4 lbs. at their free ends. Find graphically the distances of the ring from the three pulleys (see p. 56).

66. A straight uniform lever ABC can turn about a frictionless horizontal axis through its centre B. A weight of 100 lbs. is hung from A. What horizontal force applied at C will keep it in equilibrium at an inclination of 3° to the vertical?

What is the pressure on the fulcrum?

67. A span of telegraph wire weighs 14 lbs., and its inclination to the horizontal at each end is 8°. What is the tension at its ends?

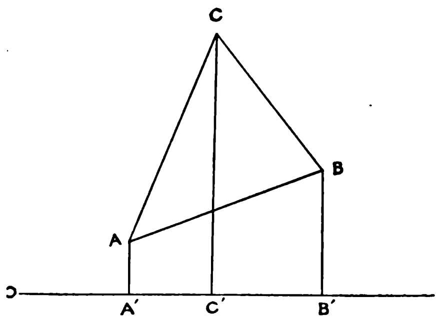
68. A punching ball, diameter 12 inches, whose weight may be neglected, is tied to the floor and ceiling by equal elastic strings each 4 feet long. It is drawn aside and held with its centre 2 feet from its original position, by a horizontal force of 50 lbs. Draw a diagram to scale and find the tensions in the elastic strings. Check by calculation.

Relations between Forces deduced from Relations between the Lines which represent them.—Lami's theorem, and the formulæ we have obtained for the resultant of two given forces, and for the resolved part of a given force in a given direction are particular cases of a general proposition, the truth of which follows from the possibility of representing a force by a straight line. The proposition may be stated as follows—

From any geometrical or trigonometrical theorem connecting the lengths of lines in a plane figure may be

inferred a proposition connecting forces represent these lines.

Thus Lami's theorem is the translation into a st proposition of the trigonometrical proposition that i triangle the sides are proportional to the sines c opposite angles.

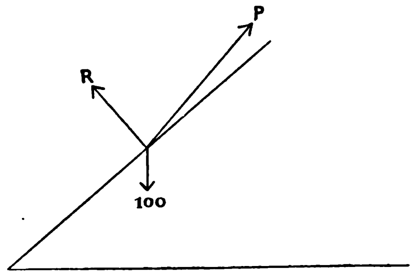


Resolving.—If ABC is any Δ and AA′, BB′ and C drawn perpendicular on any straight line OX, then A B′C′ + C′A′ = 0, or in words, the sum of the projecti AB, BC and CA on any straight line is zero; or agai sum of the projections of AB and BC is equal to the p tion of AC. We perceive that if AB represents a forc A′B′ represents the resolved part of that force in the tion OX, and that the statical analogues of the prec theorems are: (1) If three forces acting at a point equilibrium the (algebraic) sum of their resolved pa any given direction is zero; and (2) the algebraic s the resolved parts of two forces in any given direc equal to the resolved part of their resultant in that dir

It sometimes happens that we do not require a *co*

solution of a statical problem. Thus, consider the following example—

What force, applied in a direction making an angle of 50^{c} with the horizontal, will support a body weighing 100 lbs. on a smooth plane whose inclination is 40° ?



The forces acting are shown in the diagram.

We do not want to know R. The resolved parts of the forces in any direction vanish. But R has no resolved part along the plane. If, therefore, we equate the sum of the resolved parts along the plane to zero we shall have an equation not containing R. This equation is—

$$P \cos 10^{\circ} - 100 \cos 50^{\circ} = 0,$$

whence $P = 65{\cdot}4$ lbs.

The foregoing process is a very important one.

Note that we commence by specifying the *complete* set of forces in action. This step can never be omitted without great risk of error. We then choose a direction at right angles to that of a force which we do *not* require, and "resolving" in this direction we obtain an equation in which the objectionable force does not occur.

EXAMPLES.

69. If a string passes over a smooth pulley and its ends are fastened to an object, such as a rod, which is supported by it, prove that

in the position of equilibrium the two portions of the string must be equally inclined to the horizontal.

[Hint: Resolve horizontally for the forces acting on the rod.]

70. Apply Example 69 to solve the following problem by graphic construction—A rod, not uniform, 5 feet long, has a string 7 feet long attached to its ends by which it is hung over a smooth fixed pulley. It is in equilibrium when the parts of the string on either side of the pulley are 4 and 3 feet. Find the tension of the string if the rod weighs 3 lbs. Find also the inclination of the rod and the depth of its centre of gravity below the pulley.

71. A wooden cylinder, radius 4 inches, weighing 5 lbs., is hung up by a string 4 inches long, which is tied to a nail in a smooth wall, and the cylinder rests with its axis horizontal against the wall. Find the tension of the string.

[Draw the figure, mark the directions of the forces, calculate the angle the string makes with the wall, and resolve vertically.]

72. A weight of 200 lbs. is slung up by two cords which make angles of 10° and 5° with the vertical. Find their tensions by resolving perpendicularly to each cord in turn.

73. A rope ABCD, considered weightless, has its ends A and D fastened. A weight of 3 cwts. is attached to B and a weight of 5 cwts. to C. The inclination of AB to the vertical is 48°, and the tension in it is 6 cwts. Find the tension in CD and its inclination to the vertical.

74. A rod of weight W, whose centre of gravity divides it in any ratio, is supported by strings joining its ends to a point. If each string makes an angle a with the vertical, show that the tension in each string is the same, and express it in terms of W and a.

75. A string 10 feet long carries a smooth ring weighing 8 lbs., which can slide freely on the string; the extremities of the string are attached to two fixed points in a horizontal plane, distant 6 feet apart. Find the tension in the string.

76. A light string ABCD, 16 inches long, is fastened to two fixed points A, D in the same horizontal plane 12 inches apart; weights of 8 lbs. each are suspended from the string at B and C respectively. Given that AB = CD = 5 inches, and that BC is parallel to AD, find the tensions in the different parts of the string.

If a third 8-lb. weight were suspended from the middle point

of BC, prove that the tangent of the inclination of either part of BC to the vertical would be three times that of AB or CD.

77. AB, BC are two strings tied to fixed points A, C, and carrying a weight of 10 lbs. at B. If AB and BC are inclined at 36° and 51° to the horizontal, find the tensions, by resolving at right angles to AB, and then at right angles to BC.

78. A weightless string of length $3l$ has its ends fastened to two points in a horizontal line at a distance $2l$ apart, and holds two equal weights W at distances l from the ends and from each other. If T_1 be the tension of the two upper parts of the string, and T_2 the tension of the lower part, show that

$$T_1 = 2T_2 = \frac{2\sqrt{3}}{3} \times W.$$

If the string will break when the tension is $2\frac{3}{5}$ of W, show that the greatest length to which the distance between the ends of the string can be increased is $\frac{37}{39}$ of the whole length of the string.

79. A pulley B rests on a smooth horizontal plane. A string tied to a fixed point at A passes over B, then round a pulley C, whose centre is fixed, round B again and back round C, and carries a weight W at its lower end. Assume that the tension in the string is W throughout. Specify the forces acting on the pulley B, and by resolving horizontally obtain an equation connecting the angles which the strings AB and BC make with the vertical.

[Mach's apparatus for illustrating the law of refraction of light.]

80. AB, BC, CD are any number of strings knotted together and hung up from the ends A and Z. Weights are hung from the knots B, C, D, . . . Y. Prove by resolving horizontally for the equilibrium of each separate knot that the tension in any string varies inversely as the cosine of its inclination to the horizontal.

81. Two equal smooth spheres of weight 10 lbs. each and diameter 2 feet are connected by a string 25 inches in length, which is laid symmetrically over two pegs A, B in the same horizontal line and 12 inches apart. Find the tension of the string and the pressure between the spheres in the position of equilibrium.

82. A uniform rectangular plate, ABCD, of weight W can turn freely about the highest point A, which is fixed, and the middle point of BC rests on a smooth rail, so that BC is inclined at an angle α to the horizontal. *Draw* a sketch showing the forces acting on

the plate and show that the pressure on the hinge at A is $\frac{W \sin \alpha}{\cos \beta}$, where β is the inclination of AC to BC.

83. AB and BC are two equal bars jointed at B. The point A is fixed, but C can move horizontally along AC. A weight of 10 lbs. is placed on B, when the angle ABC is 120°. Find the horizontal pressure exerted by C, and draw a curve showing how this pressure changes as the angle ABC changes from 120° to 180°.

Finding Position of Equilibrium.—In certain statical problems we desire to find the *position of equilibrium*, and we do not wish to find the actual magnitudes of the forces involved. In such a case, if the number of forces acting on the body is three, the position of equilibrium can usually be found by purely geometrical reasoning, from the fact that the lines of action of these three forces must be either concurrent or parallel.

The solution *can* be found from the statical equations connecting the forces, but it is neater, and sometimes easier, to look at the matter from the above geometrical point of view, though, as will be seen, the problems thus dealt with are for the most part examination conundrums.

EXAMPLES.

84. A uniform rod rests with one end on a smooth horizontal floor, and the other end tied by a string to a fixed point. Find the position of equilibrium.

Here two out of the three forces acting on the rod are vertical, namely, the weight of the rod and the pressure of the ground. Hence the third force must be parallel to the other two, and accordingly the string must be vertical.

85. A uniform rod, BD, 3 feet long rests over a smooth peg at C, and a string is tied to the end B and fastened to the ground at A

a point vertically below C. Given AC = AB = 2 feet, find the position of equilibrium.

Notice that the vertical through the middle point G of the rod, the perpendicular to the rod at C, and the line BA must meet in a point, say O. Hence it is easily seen that the triangle OBG is isosceles, that C bisects BG, and that the rod is inclined to the horizontal at an angle whose sine is $\frac{3}{16}$.

86. A uniform rod AB, length 20 inches, is tied by a string BC, length 30 inches, to a point C in a smooth vertical wall. The end A of the rod rests against the wall at a point vertically below C. Find the position of equilibrium, showing that if $\angle BCA = \alpha$ and $BAC = \theta$ then $3 \sin \alpha = 2 \sin \theta$, and also $2 \tan \alpha = \tan (180° - \theta)$.

Find α and θ graphically, and the distance of A below C.

87. A uniform rod AB rests over a smooth peg C and against a smooth vertical wall BD. Find the inclination of the rod to the horizontal in the position of equilibrium.

88. A smooth cylinder of radius r is fixed, with its axis horizontal, against a smooth vertical wall. The ends of a rod of length $2a$ rest against the wall and the cylinder. If θ is its inclination to the horizontal and α the inclination to the horizontal of the radius to the point of contact show that $r(1 - \cos \alpha) = 2a \cos \theta$, and that $\tan \alpha = 2 \tan \theta$.

If $r = 10$, $a = 15$, find θ and α.

89. A uniform rod of length $2a$ rests at an inclination θ partly inside and partly outside a smooth hemispherical bowl of radius r, show that $2r \cos 2\theta = a \cos \theta$.

[Hint: If G is the middle point of the rod and O the point of concurrence of the lines of action of the forces, find the angles of the triangle BGO and show that $BO = 2r$.]

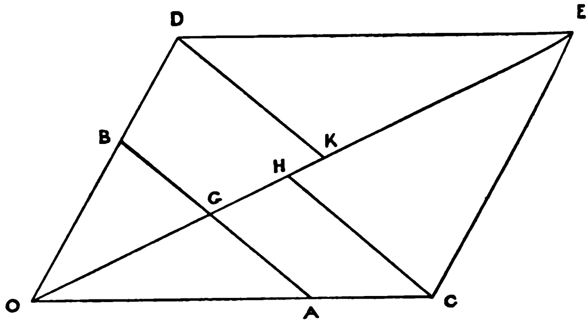
Let two forces P and Q act at a point O, and be represented by mOA and nOB respectively, where n and m are numbers. Thus we might have $m = 2$ and $n = 3$.

Produce OA to C, making OC = mOA,

" OB to D, " OD = nOB.

Complete the parallelogram OCED and join OE, cutting AB in G.

Draw CH, DK parallel to AB, meeting OE in H and K respectively.



The triangles OCH and DEK are congruent.

Therefore $\qquad \text{OH} = \text{EK},$

hence $\qquad \text{OK} = \text{EH}.$

But $\qquad \text{OH} = m\text{OG},$

$\qquad \text{OK} = n\text{OG};$

$\therefore$ adding $\text{OE} = (m + n)\,\text{OG},$

i.e. the resultant is represented by $(m + n)$ OG.

Again, $\qquad \text{CH} = m\text{AG},$

$\qquad \text{DK} = n\text{BG}.$

But $\qquad \text{CH} = \text{DK},$

$\therefore m\text{AG} = n\text{BG}.$

i.e. the point G divides AB in the ratio $n : m$.

EXAMPLES.

90. The components parallel to OG and GA of the force P are represented by mOG and mGA respectively. Replace Q by its components parallel to OG and GB, and hence prove the above

proposition, viz. that forces represented by mOA and nOB have a resultant represented by $(m + n)$ OG, where G divides AB, so that—

$$m\text{AG} = n\text{GB}.$$

[The formulæ in this example do not involve any angles, and hence are useful in such cases as the following, where length measurements are both convenient and necessary for the purpose of constructing the object.]

91. A force R acting along OG has for its components along any two lines OA and OB (ABG being in a straight line) two forces P and Q such that—

$$\text{P} = \frac{\text{OA}\,.\,\text{BG}}{\text{OG}\,.\,\text{AB}}\,.\,\text{R},$$

$$\text{Q} = \frac{\text{OB}\,.\,\text{AG}}{\text{OG}\,.\,\text{AB}}\,.\,\text{R}.$$

92. A weight of 570 lbs. is supported by two poles AO, BO, whose lower ends rest on a slope AGB. By measurement it is found that—

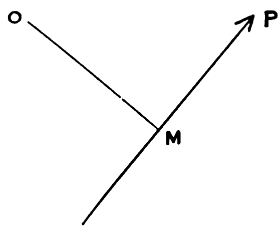
AO = 13 feet,
OB = 9·75 feet,
AB = 16·25 „

Height of O over the point G in the plane vertically below it = 8·45 feet; AG = 7·15 feet. Find the thrusts in each of the bars AO and BO.

93. Repeat experiment 1, page 74, and take the measurements necessary in order to apply the results of the preceding example.

CHAPTER IV

Moments.—We have already defined the *moment* of a force about a point as the product of the magnitude of the force with the length of the perpendicular from the point on the line of action of the force. In symbols the moment of P about O = P × OM.



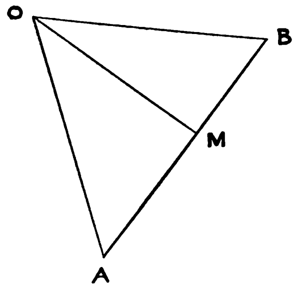
Units.—If forces are measured in pounds and distances are measured in feet the moment of a force about a point will be so many pounds-feet.

There is no objection to measuring forces in tons or distances in inches, but if we are to compare the moments of two forces we must use the same units, both of force and of distance.

We shall usually speak of the moment of a force as so many *pounds-feet*, to avoid confusion with the unit of work, which is always called a *foot-pound*.

Graphical Representation of a Moment.—If AB repre-

sents the force P then the moment of P about O is represented by AB × OM. But this is proportional to the



area AOB. Hence the moment of P about O may be represented graphically by the area AOB.

EXAMPLES.

1. If the scale for forces is 1 inch represents 1 lb., and if OM is measured in inches, upon what scale does the triangle AOB represent the moment of P about O?

2. A tall vertical post is built into the ground. A rope 20 feet long is fastened to the post and to a peg in the ground, and its tension is 200 lbs. Plot a graph showing the relation between the height at which the rope is attached and the moment of the tension about the foot of the post.

How would you attach the rope if your object were to pull the post over?

EXPERIMENT 1.

Take a circular wooden disc[1] about 14 inches in diameter and about $\frac{1}{2}$ inch thick and mount it so as to turn

[1] A ready-made disc and axle can be procured from Messrs. Cussons and other makers.

freely on an axis through its centre perpendicular to its plane. Ascertain first that the disc can turn freely about its axis and that it balances in any position. If the disc is out of balance it may be corrected by one of the methods indicated at p. 241. The axis may now be fixed in a horizontal position.

With the aid of a circular protractor[1] cut a circle out of paper and mark divisions corresponding to degrees on its edge. Paste this circle on the front of the disc so that its centre coincides with the centre of the disc. If the disc is home made the paper should be pasted on before the hole for the axle is cut. If the hole for the axle has already been cut the proper mode of securing that the disc and paper are concentric will be a problem in geometry for the student. At the same time paste a similar circular piece of paper on the back of the disc to prevent the disc from warping.[2]

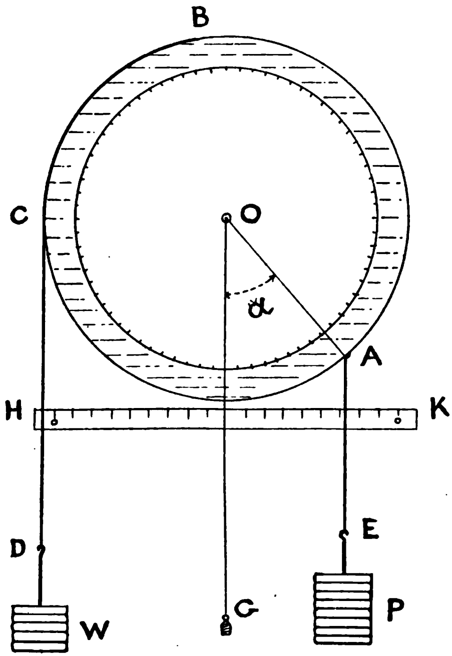
Put two tacks A and B about 150° apart on the rim of the disc, and to these attach threads which will bear a tension of at least 2 lbs. From one of these threads, BCD, hang a weight W of about $\frac{1}{2}$ lb. so that the part BC of the thread lies along the edge of the disc, and from the other, AE, hang weights of 0·5, 0·6, 0·7, 0·8, 1, 1·2, 1·6, 2 lbs. in succession.

[1] Paper graduated with radial lines at 5° intervals may be procured from Messrs. Bemrose. Alternatively, the radius of a circle in which a chord $\frac{1}{2}$ inch long subtends 5° at the centre may be calculated, the circle drawn and the $\frac{1}{2}$ inch chords stepped off with dividers, or section paper pasted round the rim.

[2] A piece of wood even 1 inch thick may warp if a piece of paper *is pasted* on one side only.

Hang a small weight by a thread from the axle O to act as a plumb line.

The angle α which OA makes with the thread can thus be read off for each value of P.



Fix a ruler HK horizontally below and just clear of the disc.

Tabulate your results thus—

Weight P.	Angle α.	Distance x.	Moment of P about O.

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What is the moment of W about O?

Is the moment of P about O constant?

Draw on squared paper a curve showing the relation between P and x.

Show that if the apparatus were perfect the relation between P, W and α would be $P = W \operatorname{cosec} \alpha$.

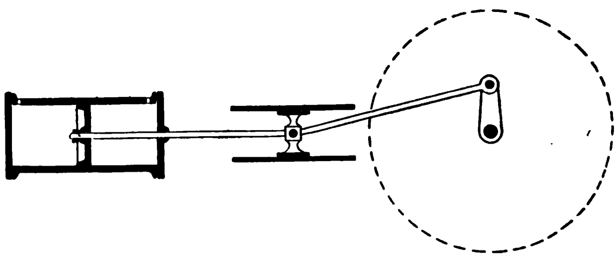
Test your own results by comparing the value of $\frac{W}{P}$ with $\sin \alpha$.

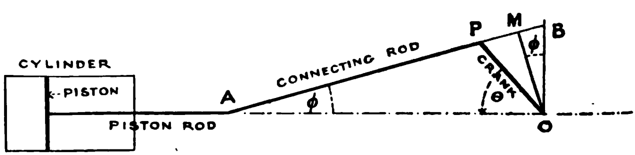
Turning Moment on a Crank.—

Let P be the thrust of the steam on the piston (see p. 243).

S the thrust in the connecting rod.

Y the lateral forces due to the constraints which keep the piston and rod moving in the line AO.





If we assume that the piston rod is in equilibrium[1]

[1] This investigation is only approximate, because in reality, first, the piston is resisted by the friction or rubbing against the cylinder

under these forces, and that the forces Y act perpendicularly to OA, we have, resolving parallel to OA—

$$S \cos \phi = P.$$

Draw OB at right angles to AO, meeting AP produced in B, and drop the perpendicular OM on AB. The moment about O of the thrust S = S × OM = S . OB cos ϕ = P × OB. So OB represents graphically the turning moment due to a constant thrust P.

EXAMPLES.

3. Set off a series of positions of the crank,[1] say at every 10°, and set off along each such position OP a line OQ equal in length to OB. The locus of Q gives a "radial" curve of turning moment.

Notice that—

$$OB \cos \phi = OM = OP \sin \overline{\theta + \phi},$$

so that if we wish to calculate the turning moments we have, putting AB = b, OP = a— ;

$$b \sin \phi = a \sin \theta. \qquad (1)$$

$$\text{Turning moment} = \frac{Pa \sin \overline{\theta + \phi}}{\cos \phi}. \qquad (2)$$

4. Given $a = 1$, P = 1, $b = 4$, calculate the turning moment when $\theta = 10°, 20° \ldots 180°$, and tabulate the results.

5. Show that the turning moment is the same for $\theta = 180° + a$ and $\theta = 180° - a$.

Hence complete the radial curve of turning moments from 180° to 360°.

6. Using the foregoing results, draw the radial curve of turning moments for two cranks set at 90° from one another, setting off the

---

and guides, and secondly, the piston is in reality moving with varying speed, and so the forces acting on it are not really in equilibrium (see p. 2). The errors introduced by our assumption are, however, in many cases small.

[1] The radial paper mentioned at p. 92 is useful for this purpose.

line representing the total turning moment along the line showing the position of the first crank.

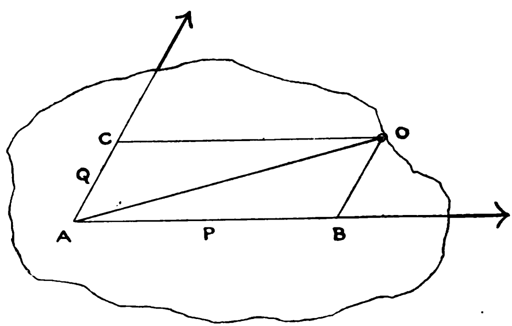
[It will be observed that there is now much less variation in the turning moments.]

7. Using the foregoing results, draw the radial curve of turning moments for three cranks set at 120° to one another.

These examples should be worked out by a class in conjunction, each student working out results for one or two values of θ.

The effect of varying the length of the connecting rod may also be studied in a similar manner by taking $b = 1{\cdot}5$ and $b = 10$.

Body turning about a Fixed Point.—Suppose a body which can turn about a fixed point O to be in equilibrium when acted on by two forces P and Q. Replacing P and Q by their resultant R, we see that the line of action of the



resultant must pass through O, or the body will turn round O and not remain in equilibrium.

Let the lines of action of P and Q meet in A.

Join OA and construct the parallelogram ABOC, whose

sides are parallel to the directions of P and Q, and whose diagonal is OA.

If then AO represents R,

AB will represent P,

AC " " Q,

and $\triangle ABO = \triangle ACO$;

that is, the moment of P about O must equal the moment of Q about O.

If then a body can turn about a fixed axis which meets the plane of the forces in O, the condition of equilibrium is that the moments of P and Q about O shall be equal in magnitude, but opposite in "sense," that is, way of turning.

It might seem that, however great the forces P and Q, equilibrium will exist if the foregoing condition is satisfied. But all that the foregoing statement asserts is that if the axis remains fixed the body will not move as a whole. The questions whether the axis is sufficiently strong, or whether the body will be distorted or broken by the forces, are not dealt with at present.

Bent Lever.—The conclusion just arrived at shows that if a lever, whether straight or bent, whose fulcrum is at O, is kept in equilibrium by two forces P and Q, the moments of P and Q about O must be of opposite senses, but equal in magnitude.

EXAMPLES.

8. ACB is a straight lever, C is the fulcrum, AC = 9 inches, CB = 4 inches. What is the greatest weight that can be lifted by means of the lever, if the pressure on the fulcrum is not to exceed 26 lbs.?

9. A straight but not uniform lever balances 6 inches from one

end. If when supported at 8 inches from that end it requires 3 lbs. at the other end to balance, and when supported at 10 inches from the first end it requires 8 lbs. at the other: find the length of the lever.

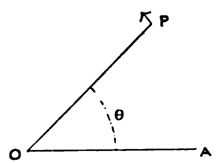
10. A weight A of 12 lbs. and a weight B of 18 lbs. balance on a straight lever, whose weight may be neglected. If their positions are interchanged, find (1) the weight which must be added to A, (2) the weight which must be subtracted from B, to produce equilibrium.

11. A uniform bar in the shape of a rectangle ABCD weighing 10 lbs. is hung up by the middle point of AB, through what angle will it be deflected by suspending 1 oz. from A? AB = 4 feet; BC = 1 inch.

12. A tradesman weighs out his goods with a straight uniform lever having each arm 12 inches long, but falsifies it against the customer by shifting the fulcrum $\frac{1}{4}$ inch. If the apparatus weighs 1 lb., prove that he cheats the customer of $\frac{16}{49}$ oz. on every false pound sold.

13. A bar AOB, 12 inches long and weighing 0·17 lb., turns about O, which is midway between A and B. The centre of gravity G is 0·23 inch below O, and OG is perpendicular to AB. What angle does OG make with the vertical when a weight of 1 lb. is hung on A and 1·001 lbs. on B?

+ and − applied to Moments.—If a force P tends to turn a body round O in the sense of the hands of a watch, say "clockwise," then a force Q tending to produce rotation in the opposite sense, say counter-clockwise, will *oppose* P, and the turning effect will be measured by the *difference* of the moments. We may conveniently use + and − signs to distinguish between the moments of forces in opposite senses.



In trigonometry we always consider the sense of counter-clockwise rotation as positive, and it will be convenient to do the same here.

It may perhaps be regarded as obvious that *the sum of the moments of two forces about any point is equal to the moment of their resultant about that point*, the word sum meaning, of course, "algebraical sum," so that if M_1, M_2 are the moments of the forces, we have—

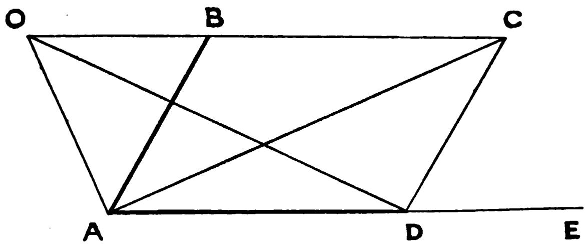
$$M_1 + M_2 = \text{moment of resultant},$$

and if $$M_1 = 7 \text{ and } M_2 = -5,$$

then $$(+7) + (-5) = \text{moment of resultant},$$

i. e. moment of resultant $= 2$.

It is, however, of interest to verify that the proposition follows from the parallelogram of forces.



Let forces P and Q act at A along AB and AE respectively. Draw through O a line OB parallel to AE.

As we may represent P and Q on any scale we please, let us take AB to represent P, and on this scale let AD represent Q.

Complete the parallelogram ABCD.

The moment of P about O is represented by the Δ AOB,

" " Q " O " " " Δ AOD.

But $$\Delta\,AOD = \Delta\,ACD = \text{half parallelogram},$$
$$= \Delta\,ABC.$$

The sum of the moments of P and Q about O is represented by Δ OAB and Δ ABC, that is, by Δ OAC.

But AC represents the resultant of P and Q, and consequently the triangle OAC represents the moment of the resultant about O.

That is, the sum of the moments of P and Q about O is equal to the moment of their resultant about O.

EXAMPLES.

14. After the manner of the foregoing reasoning consider the cases (*a*) when O is within the angle BAE, (*b*) when O lies on the line AE.

15. Draw lines AB, AD representing P and Q on any scale. Draw OEF parallel to BD, meeting AB in E and AD in F. Take AE, AF to represent P and Q on a new scale, and bisect EF in G. The triangles OAE, OAF, OAG represent the moments of P, Q and their resultant about O. Deduce the proposition.

The case in which P and Q are parallel will be dealt with subsequently.

Cor. 1. The algebraical sum of the moments of any number of forces about a point O in their plane is equal to the moment of their resultant.

Cor. 2. If any number of forces in a plane have a single resultant the sum of their moments about any point on the line of action of the resultant vanishes.

Cor. 3. If a number of forces in a plane are in equilibrium the sum of their moments about *any* point in the plane vanishes.

EXAMPLE.

16. *Applications of Cor.* 2. Let three forces of 6, 5 and 7 lbs. act along the sides of an equilateral triangle ABC, sides 1 foot, as in *the figure.* To find the line of action of their resultant.

Suppose the line of action of the resultant cuts AB in the point O.

Since the sum of the moments about O vanishes,

$$7OM = 5ON,$$

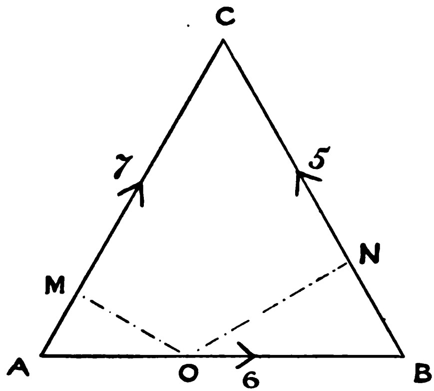
$$i.\,e.\ 7 \times AO \sin 60° = 5OB \sin 60°,$$

$$\therefore \frac{AO}{OB} = \frac{5}{7},$$

or

$$AO = 5 \text{ inches},$$

$$OB = 7 \quad ,,$$



If the line of action of the resultant cuts BC in Q, then, since the sum of the moments about Q vanishes,

therefore $\qquad 7 \times CQ \sin 60° = 6BQ \sin 60°,$

or

$$\frac{CQ}{BQ} = \frac{7}{6},$$

$$i.\,e.\ \frac{CQ}{12 - CQ} = \frac{7}{6},$$

whence

$$CQ = 6\tfrac{6}{13} \text{ inches},$$

$$BQ = 5\tfrac{7}{13} \quad ,,$$

17. Forces of 1, 2, 3 and 4 lbs. act consecutively along the sides of a square, of 1 inch side. Find the position of the line of action of the resultant.

18. A triangular lamina ABC can turn about a point O in AC. Forces of 10 lbs. each act along BC, CA, BA respectively, but do not move the lamina. Find the distance AO, if CA = 3 inches CB = 4 inches, AB = 5 inches.

19. ABC is an equilateral triangle, and forces P, P and 2P act along BA, AC and CB respectively. Find the magnitude of the resultant and the point at which it cuts the line AC produced if necessary.

20. If D is the middle point of the side BC of an equilateral triangle ABC, and if forces of 10, 1, 4 and 3 lbs. act respectively along BA, AD, CA and CB, find the position of their resultant, given that BC = 2 feet.

21. Forces P, Q, R act along the sides AB, BC, CA of any triangle, and have a single resultant whose line of action cuts AB in D, BC in E and CA in F. Take moments about D, E, F in turn, and hence show that AD, BE, CF = DB, EC, AF. [This theorem is attributed to Menelaus, circa 80 A.D.]

Application of Cor. III., p. 100.—The fact that if any number of forces keep a rigid body in equilibrium, the algebraic sum of their moments about any chosen point will vanish, furnishes the simplest means of dealing with most problems on parallel forces.

EXAMPLES.

22. A uniform bar AB, weighing 20 lbs., 5 feet long, rests horizontally on supports at its ends A and B. Find the vertical reactions at these supports if the bar is loaded with weights of 30, 40 and 50 lbs. at points 1, 2 and 3 feet from A.

Let P and Q be the vertical reactions at A and B. The sum of the moments of all the forces about B vanishes, therefore—

$$30 \times 4 + 40 \times 3 + 20 \times 2{\cdot}5 + 50 \times 2 - 5P = 0,$$

or

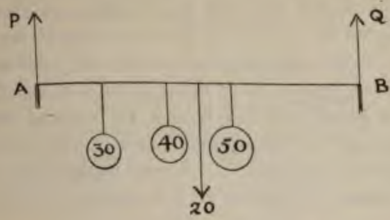
$$P = 24 + 24 + 10 + 20 = 78 \text{ lbs.}$$

The sum of the moments about A vanishes, therefore—

$$5Q - 50 \times 3 - 20 \times 2{\cdot}5 - 40 \times 2 - 30 \times 1 = 0,$$

or

$$Q = 30 + 10 + 16 + 6 = 62 \text{ lbs.}$$



As a check, observe that, resolving vertically, we must have

$$P + Q - 30 - 40 - 20 - 50 = 0,$$

or

$$P + Q = 140.$$

Write down the algebraical sum of the moments of the system of forces about a point in the beam distant x from A.

23. Two men support the ends of a uniform rod whose length is 6 feet and whose weight is 2 lbs., from which are hung weights of 10 and 12 lbs., 2 feet from the ends. What weight does each man carry?

24. A uniform rod of length 6 feet and weight 4 lbs. has weights of 1, 2 and 3 lbs. attached 1, 2 and 4 feet respectively from one end. At what point must it be supported in order that it may be in equilibrium?

25. A heavy uniform beam, 30 inches long and weighing 24 lbs., rests on supports, one of which is at one end and one 6 inches from the other end. Find the pressures on the supports.

If an additional weight of 6 lbs. is hung from the beam between the centre and the second support, find its greatest distance from the centre if this support breaks with a stress of 20 lbs.

26. On a bridge of 80 feet span two loads are standing; one of 3 tons at a place 10 feet from one end of the bridge, and another of 8 tons 20 feet away from the first. What are the pressures on each of the supports of the bridge, and how far is the centre of gravity of the loads from each end?

27. A uniform heavy rod balances about a point 1 inch from its centre when weights of 3 and 5 lbs. are hung from its extremities; if it balances about the same point when these weights

are replaced by weights of 1 and 2 lbs. respectively, find its length.

28. A heavy rod (not uniform) balances about a point in its length with weights of 6 and 12 lbs. respectively hung from the two points of the rod A and B; and it also balances about the same point when weights of 9 and 20 lbs. are respectively hung from A and B. Show that it also balances about the point when weights of 15 and 36 lbs. are respectively hung from A and B.

29. A uniform plank ABC of length 14 feet and weight 90 lbs. rests horizontally on two supports at A and B, one at the end A and the other $2\frac{1}{2}$ feet from the end C. A man walks from A to C along the plank, and just as he reaches C it commences to tilt. Find his weight.

30. A straight uniform rod, weighing 1 lb. and measuring 1 yard in length, lies on a horizontal table with 2 feet of its length projecting over the edge of the table. What least weight on the inner end, or what least support at the outer end, would maintain equilibrium?

31. A uniform bar AB, 5 feet long, weighing 120 lbs., is hung up by two vertical strings, one attached at A, the other at a point 2 feet from B. What weight, hung from B, will just cause one string to become slack? What will then be the tension of the other string?

32. A cube of stone, edge 3 feet, weighs 5000 lbs., and rests on one of its faces. A lever 6 feet long can be pushed 6 inches underneath the lower face in a direction at right angles to an edge. What upward force must be exerted by a man at the end of the lever to raise this edge from the ground?

33. A uniform beam AB of length 8 feet rests horizontally on two supports 1 and 5 feet from the end A. If the greatest weight which can be hung on the end B without upsetting the beam is 5 lbs., find the weight of the beam and the greatest weight which can be hung at A without upsetting the beam.

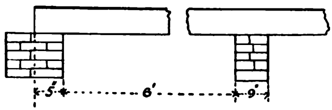
34. A uniform steelyard is 2 feet long and weighs 10 lbs. The pan weighs 35 lbs., and is hung from one end. The fulcrum is 3 inches distant from that end. A weight of 5 lbs. slides on the beam, which is graduated to read to $\frac{1}{4}$ lbs. Find the distance between the graduations, and also the greatest weight which the balance can weigh. [Army]

35. An unloaded steelyard hung about F balances when the

movable weight P is at O, and about F′ when the movable weight is at O′. Prove that the weight of the steelyard $= -P + P \cdot \frac{OO'}{FF'}$.

36. A uniform beam weighing 6 lbs. and 4 feet long has weights of 5 and 3 lbs. suspended from its ends. On what point will it balance?

37. An iron bar 10 feet long, weighing 105 lbs., rests in a horizontal position on two walls as in the diagram. Determine the



pressure on the walls supposing it to be uniformly distributed at each wall.

38. A rectangular column, 20 feet high and 4 feet square in section, is not fixed to the ground but merely rests on it. What force acting perpendicular to a face at its centre would overturn it, the material weighing 150 lbs. per cubic foot?

39. A uniform square lamina can turn freely in a vertical plane about a horizontal axis through its centre of gravity, and forces of 1, 2, 3, 4 lbs. weight respectively act along its edges in such directions that all tend to make the square rotate in the same direction. Find the weight of a heavy particle which, being attached to one corner, will keep the square in equilibrium with its lowest edge horizontal.

40. A uniform girder, 39 feet long, weighing 12 tons, rests on two masonry piers. If the centres of the piers are at distances 5 and 10 feet from the ends of the girder, what weight is supported by each?

41. A uniform rod, weighing 60 lbs., rests symmetrically upon two supports in a horizontal position. A weight of 30 lbs. is placed upon it 4 feet from its centre. What other weight must be placed on it, and where, in order that the pressure on each support may be 57 lbs.?

42. A steelyard 20 inches long, and weighing 2 lbs., is divided along 18 inches from one end, the divisions running from 0 to 18, each division corresponding to 1 lb. when the running weight is 1 lb. Prove that the fulcrum is 1 inch from the zero of the divisions and $\frac{1}{2}$ inch from the centre of gravity of the steelyard.

43. The beam and scale-pan of a common steelyard weigh $1\frac{3}{4}$ lbs., and their combined centre of gravity is $1\frac{1}{2}$ inches from the fulcrum on the same side as the scale-pan. The distance of the scale-pan from the fulcrum is 4 inches and the zero mark is $2\frac{1}{2}$ inches from the fulcrum on the other side. If the greatest load that can be weighed is 6 lbs., find the magnitude of the travelling weight and the length of the graduated part of the arm.

44. A ladder 18 feet long has its centre of gravity 7 feet from one end. It is carried by two men, one taking hold 4 feet from the heavy end, the other 6 feet from the light end. How is the weight distributed between them?

45. A uniform rod, of length 6 feet, weighing 10 lbs., rests with its extremities on two props A, B; a second rod, of length 6 feet and weight 8 lbs., rests with one extremity on a prop C, and 2 feet of its length extending across the rod AB, which it intersects at a distance of 1 foot from A. Find the total pressures on the three props.

46. A heavy uniform ladder, weighing 120 lbs. and 20 feet long, is carried on the shoulders of two men respectively at distances 3 and 5 feet from the ends. Find the pressures on their shoulders.

47. A railway carriage whose weight is 24 tons is exposed to a wind pressure of 10 lbs. per square foot acting upon one of its sides. The area exposed to wind pressure is 500 square feet, and the resultant pressure acts at a height of 8 feet above rail level. If the distance between the rails is 4 feet $8\frac{1}{2}$ inches, what is the vertical reaction on each rail? Determine also the least wind pressure that will overturn the carriage.

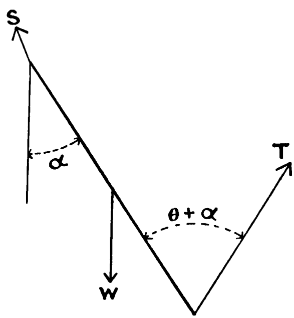
48. A heavy rod passes through two fixed rings which are 3 feet apart and in the same horizontal line. The pressures on the rings are P and Q, while if the rod is pushed on 6 inches through the rings, the pressures are Q and P. Find P and Q in terms of the weight of the rod.

"**Taking Moments.**"—We have already noticed that the magnitude of a single force may sometimes be very conveniently found, without effecting the complete solution of a statical problem, by *resolving* in some particular direction. We can often obtain all the information we need, as

regards some particular problem, from an equation expressing the fact that the sum of the moments of the forces in action about *one* judiciously-chosen point vanishes.

EXAMPLES.

49. Consider a heavy uniform rod AB, length $2a$, weight W, which can turn about a horizontal axis at A, and is pulled aside by a string tied to the end B. Given the inclination of the rod to the vertical $= \alpha$, and the inclination of the string to the vertical $= \theta$, find the tension of the string.



The forces acting on the rod are shown in the figure. We do not require the force S, so we will take moments about A, so as to obtain an equation not involving S. This equation is—

$$T \,.\, 2a \sin \overline{a + \theta} - W \,.\, a \sin a = 0,$$

or

$$T = \frac{W \sin a}{2 \sin (a + \theta)}.$$

50. A heavy uniform rod, 12 feet long, weighing 120 lbs., rests with one end on a horizontal plane, and can turn about a horizontal axis at right angles to the rod and 2 feet above the plane. If the rod projects 2 feet beyond the axis, find the force at the upper end which, acting on an angle θ with the rod, will just make it begin to turn.

[If the rod has just commenced to turn, the lower end is just off the ground.]

51. A force of 10 lbs. acts tangentially at the circumference of a pulley 2 feet in diameter. What force must be applied to a rope coiled round the shaft to which the pulley is keyed (or fixed) to

prevent the pulley from turning? The diameter of the shaft is 3 inches. [Army]

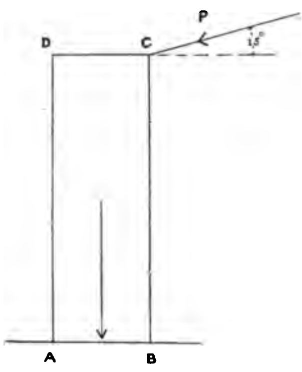
52. A cubical case, side 2 feet, is on the floor. A man pulls at the right-hand top edge so as to turn the case over about the left-hand lower edge. Find the least pull in a direction inclined at θ to the horizontal which will just *begin* to turn the case. What is the best value of θ?

53. A garden roller weighing $1\frac{1}{2}$ cwts., radius of roller 16 inches, is to be drawn up a step 6 inches high. Assuming the pulling force to be inclined upwards at 10° to the horizontal, find the least pull which will just begin to raise the roller off the ground. What is, at this instant, the reaction of the edge of the step in magnitude and direction?

[The lines of action of the pull and of the weight may be assumed to pass through the centre of the roller.]

54. A wheel of radius R and weight W rolling along a horizontal road comes to rest against an obstacle of height h. Find the least horizontal pull X applied to the axle which will begin to raise the wheel. Find the direction of the very least pull P, applied to the axle, which would begin to raise the wheel.

Taking W = 100 lbs., h = 2 inches, work out X when R = 12, 18, 24, 36, 48 inches. Plot a graph showing the relation between R and X.



It is sometimes convenient in finding the moment of a force about a point to replace the force by its components in two directions at right angles.

55. A force P acts at an inclination of 15° on a uniform block weighing 2 cwts., as in the figure. Find the magnitude of P if it would just begin to upset the block about A, given AB = 2 feet, BC = 6 feet.

When the block is just beginning to turn it rests

on the ground at A, and so the reaction of the ground passes through A. Taking moments about A, and replacing P by P cos 15° along CD and P sin 15° along CB, we have—

$$P \cos 15^\circ \times 6 - P \sin 15^\circ \times 2 - 2 \times 1 = 0,$$

whence $P = \dfrac{1}{3 \cos 15^\circ - \sin 15^\circ} = \dfrac{1}{2{\cdot}9 - 0{\cdot}26} = \dfrac{1}{2{\cdot}64} = 0{\cdot}38$ cwt. nearly.

56. A picture weighing 40 lbs. rests with its lower edge on two nails in the same horizontal line, and a cord 24 inches long is attached to the middle point of its upper edge, and to a nail in the wall, the picture leaning forward so that its upper edge is 6 inches from the wall. If the weight of the picture acts in a line 3 inches from the wall, draw a side view of the picture and cord, and find the tension of the cord, if the distance between the upper and lower edges of the picture is 42 inches.

57. A uniform rod 10 inches long $\frac{1}{2}$ inch square, weighing $\frac{3}{4}$ lb., can turn about an axis at right angles to its length at the middle of the longer edges of the top face. Find the horizontal force applied at the middle of one of the shorter edges of the top face which will maintain equilibrium with the rod inclined at 5° to the horizontal.

58. A uniform ladder is lying on the ground with one end held down. A man raises the ladder into a vertical position by lifting up the other end and then walking along under the ladder, moving his hands from rung to rung. Find the least force necessary to support the ladder at an inclination α to the horizontal, the ladder being l feet long, weight W, and the man's hands h feet from the ground.

59. A 12-ton gun is raised by tackle at a point 4 feet 4 inches in the rear of the centre of gravity, and is supported by blocks at a point 1 foot 8 inches in front of the centre of gravity. Find how the weight is distributed between the tackle and the blocks.

60. A telegraph pole 30 feet high is supported by a wire fastened to the top and to the ground 10 feet from the bottom. If the pull in the wire is 50 lbs. weight, what is the horizontal pull on the pole, and what is the moment of the pull about the bottom of the pole?

61. A capstan (see diagram, p. 118) is moved by six equal bars,

each 8 feet long, and is worked by eighteen men, three at each bar. The man at the end of each bar pushes with a force of 80 lbs., the next man, at 2 feet from the end, with a force of 100 lbs., and the next man, at the middle of the bar, with a force of 120 lbs. What resisting force, acting 6 inches from the axis of the capstan, can they overcome?

62. Three men at the end of three bars of a capstan undertake to hold the capstan still against five others. At what distance from the centre should the five be placed to make a match, all men being supposed equally strong?

63. A capstan has an axle 2 feet in diameter, and is fitted with six bars each projecting 7 feet from the outside rim of the axle. Twelve men push, each with a force of 50 lbs., six at the six ends and six at a distance of 2 feet up the bars. Find the weight lifted by a rope round the axle.

64. A capstan is 3 feet in radius and the capstan bars project 10 feet (measured from the centre of the capstan). A man undertakes by shoving at the end of a bar to hold the capstan against six men of equal strength hauling on a rope round the capstan. Can he do it? Give your reasons.

65. A bar CD, 6 inches long, is pivoted at D. A block A is compelled by guides to travel in a straight line, the prolongation of which passes through D. A and C are joined by a bar 8 inches long, attached by pin-joints. If a thrust of 300 lbs. weight acts on A in the direction AD when A is 11 inches from D, what force acts in consequence along the bar AC? Enumerate the forces acting on the block A.

66. To support a railway signal post 30 feet high, a second post 15 feet high is put up 20 feet away. A wire from the top of the signal post is made fast to the top of the other post and then carried to the ground and made fast 10 feet from the foot of the second post. Find what the pull in each part of the wire must be to produce a horizontal pull of P lbs. weight on the signal post.

67. The points of contact with the ground of the front and hind wheels of a wagon are 3 feet 6 inches apart. The loads on the front and hind wheels are 5 and 9 cwts. respectively. The wagon rests on a platform 20 feet long, supported in a horizontal position by props at its extremities. What are the pressures on these supports due to the weight of the wagon when its hind wheels are exactly over the centre of the platform?

68. A uniform rod 12 inches long, whose weight is 20 lbs., is suspended in a horizontal position by two vertical strings, each of which can just sustain 100 lbs. What is the greatest weight that can be hung from (1) the centre of the rod, (2) a point 2 inches from the centre, without breaking a string? Assume the strings to remain vertical and the rod to remain horizontal.

69. Two pegs in the same horizontal line are 2 feet apart. A bar AB, 3 feet long, weighing 30 lbs., is laid symmetrically on the pegs, and on it is placed a precisely equal bar CD so that the end C projects 1 foot beyond A. Find the pressures on the pegs. If a gradually increasing downward pressure be applied at C, in what way will motion commence?

70. A straight horizontal bar ABCD, 13 feet long, whose weight may be neglected, rests on two supports, one at the end A, the other at C, a point 4 feet from the other end D. Weights of 50 lbs. are suspended from D and from B, which is distant 4 feet from A. What are the pressures on the supports?

71. Two heavy uniform rods CA, CB, of the same thickness and material, are rigidly jointed at C so as to form a right angle. If the body can turn freely about C, and CA is double the length of CB, show that in the position of equilibrium the tangent of the angle which AB makes with the vertical is $\frac{6}{7}$.

72. Two uniform rods AB, BC of the same metal and thickness are rigidly jointed at B at an angle of 45°. Compare the length of the rods if BC is horizontal when the body is suspended freely from A.

[This arrangement has been suggested as a rough level.]

73. A toasting-fork in which the three prongs are each 6 inches long, the distance between the prongs is 2 inches, and the handle is 18 inches long, is hung up from the outer end of an outer prong. Show that the tangent of the angle which the handle makes with the vertical is $\frac{20}{87}$, assuming the prongs and handle made of the same uniform wire.

74. A body of weight W is weighed in a balance in which the arms are of unequal length, and the scale-pans of equal weight. It appears to weigh W_1 in the right scale-pan, and W_2 in the left. Prove that the weight of a scale-pan is $\dfrac{W^2 - W_1W_2}{W_1 + W_2 - 2W}$.

75. A body is weighed in a balance whose arms are unequal, and whose scale-pans are of unequal weight, but which rests horizontal

when unloaded. Show how to find the true weight of a body which appears to weigh W_1 if placed in the right-hand scale-pan, and W_2 if placed in the left.

If a body rests against a *smooth* surface, being in contact with it "at a point," the direction of the action and reaction between the two bodies is along the line at right angles *to both surfaces* at the point of contact. This line is technically called the "Common Normal."

76. A uniform rod ABC in a vertical plane rests on a smooth rail at B, and is held in position by a string tied to A, and to a point on the ground vertically below B. If BC = 3AB, show that the length of the string must equal the height of B above the ground, and that its tension is equal to the weight of the rod.

Parallel Forces.—We have deduced a method of finding the resultant of two or more parallel forces from the principle of the lever. It is of interest to show that the same conclusions may be deduced from the principle of the parallelogram of forces.

1. Consider two parallel forces P, Q represented by the lines AB, CD (see figures opposite).

2. Imagine two equal and opposite forces X applied one at A the other at C.

3. Replace X and P by their resultant R, and X and Q by their resultant S, found by the parallelogram rule.

4. If the lines of action of R and S meet in O, we may imagine R and S to act at O.

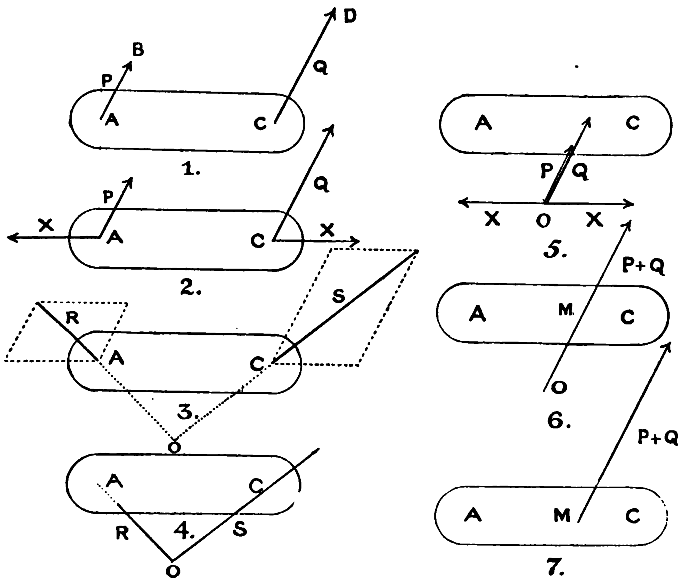
5. R at O may be decomposed into two forces X and P parallel to the original directions of X and P. Similarly, S at O may be decomposed into X and Q.

6. The two equal and opposite forces X balance one

another, and consequently the force P + Q at O is the resultant of the original forces P and Q.

7. P + Q may be supposed to act at M.

The foregoing proof depends upon the assumption:



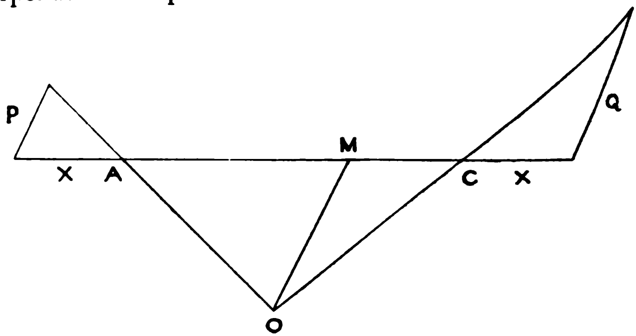
A force may be imagined to be applied to a rigid body at any point on its line of action, that is to say, it is immaterial at what point in its line of action a force is supposed to act on a rigid body (p. 73).

Example.

77. Show that if P and Q are supposed to act at any other points A′ C′ on their lines of action, the magnitude and line of action of the resultant will be unaltered.

The foregoing construction has determined the *magnitude* of the resultant and the fact that it is parallel to the origninal forces.

It remains to find how the *position* of the line of action depends on the positions of A and C.



Let the line of action of the resultant cut AC in M.

Then (1) $\frac{OM}{AM} = \frac{P}{X}$,

(2) $\frac{MC}{OM} = \frac{X}{Q}$.

Therefore, multiplying together—

$$\frac{MC}{AM} = \frac{P}{X} \cdot \frac{X}{Q}.$$

Hence the line of action of the resultant cuts AC in a point M which divides AC, so that $\frac{AM}{MC} = \frac{Q}{P}$.

Examples.

78. Draw the set of six figures and explain the corresponding construction for the case when P and Q are in opposite directions.

You will conclude that the resultant is $P \sim Q$, and that it acts at a point M in AC produced towards the side of the greater force such that $Q \times CM = P \times AM$.

Cor.—The position of M depends on the magnitude of P and Q and on their being parallel, but *not* on their having any particular direction. That is, if P applied at A and Q applied at C are turned through any angle, their resultant will still act at M, and will be turned through the same angle.

The position of M is readily found, as we have already noticed, by observing that the moments of P and Q about M are equal in magnitude and opposite in sense.

79. Examine whether the moment about any point of the resultant of two parallel forces is equal to the algebraic sum of their moments.

Cor.—If any number of given parallel forces act at points A, B, C . . . their resultant passes through a certain point O whose position does not depend on the direction of the forces.

80. Three equal like parallel forces act at the corners of an equilateral triangle. Find the point through which their resultant always passes.

[The point is sometimes termed the centre of the parallel forces.]

81. Parallel forces of 10, 18, 14 and 6 lbs. weight act at the angular points, taken in order, of a square. Find the position of the centre of parallel forces when the forces 10 and 18 act in one direction and the forces 14 and 6 in the opposite direction.

82. Four like parallel forces of 1, 2, 3 and 4 lbs. act at the corners of a square whose side is 4 inches. Find the position of the centre of these parallel forces.

83. Give a geometrical construction for the point M in which the line of action of the resultant of parallel forces P at A and Q at B cuts AB.

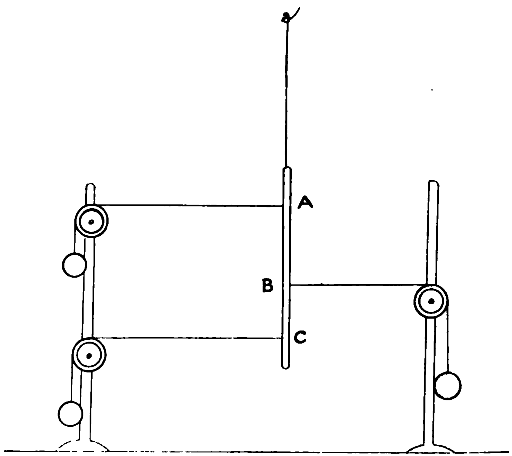
84. Unlike parallel forces, *i.e.* parallel, but in opposite directions, act at two points, A, C, 1 inch apart. Calculate the magnitude of the resultant, and the distance from A of the point in which its line of action cuts AC produced in the following cases—

(1) $P = 1$ lb., $Q = 1{\cdot}1$ lbs.
(2) $P = 1$ lb., $Q = 1{\cdot}01$ lbs.
(3) $P = 1$ lb., $Q = 1{\cdot}001$ lbs.

EXPERIMENT 2.

Hang up a straight light rod by a fairly long string. From pins inserted at points A, B, C on the rod horizontal threads pass over pulleys whose centres are fixed.

Make AB = 4 inches, BC = 3 inches, and hang weights of 0·3 and 0·4 lb. over the pulleys corresponding to A and C.



Find what weight hung over the remaining pulley B will maintain the rod vertical.

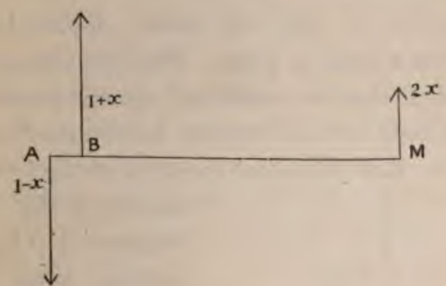
A vertical frame, to the sides of which the pulleys can be screwed, and from the top bar of which the rod can be hung, is useful for this experiment.

Does the tension of the vertical string affect the relation between the weights?

EXAMPLE.

85. Arrange the experiment so as to show the resultant of unlike parallel forces of 0·2 and 0·3 lb. acting in lines 2 inches apart.

Equal unlike Parallel Forces.—The case of two equal parallel forces in opposite directions is of great importance. Two such forces applied to a body will certainly not be in equilibrium. But if we attempt to find their resultant, by the rule for parallel forces, the conclusion reached appears to be meaningless, namely, that the magnitude of the resultant is zero, and the distance of line of action of the resultant from that of either force is infinite. This result will be, perhaps, less perplexing if it is thought of as the limit of the following. Forces



$P = 1 - x$ and $Q = 1 + x$ act in parallel lines in opposite directions 1 inch apart. The magnitude of their resultant is $2x$, while if P acts at A and Q at B, the resultant acts at M, where

$$MA(1-x) = MB(1+x),$$

so that

$$\frac{AM}{1+x} = \frac{BM}{1-x} = \frac{AB}{2x}.$$

Now let
$$x = \frac{1}{10^n},$$

and we see that the larger n is the smaller is the difference between P and Q, and the greater is the distance AM.

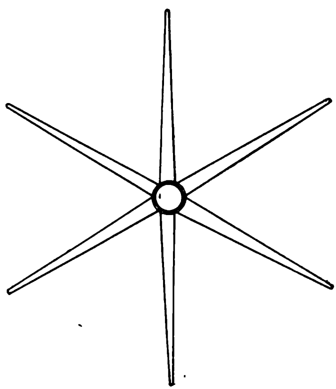
EXAMPLE.

86. Let AB = 1 foot. Find the moment of the resultant about A, B, and a series of points distant 1, 2, 3 . . . feet from B, supposing $x = \frac{1}{1000}, \frac{1}{1000000}$ . . . of a foot.

This example indicates that although the magnitude of the resultant diminishes continually as x diminishes its moment about any point near AB does not vanish, but remains approximately constant.

Couples.—To a pair of equal, oppositely-directed parallel forces a name is given. They are termed a *couple.*

It is evident that two equal and opposite parallel forces acting on a body have a tendency to turn the body, and it is easy to see that the actual direction of the forces which constitute the couple is immaterial so far as their turning effect is concerned.

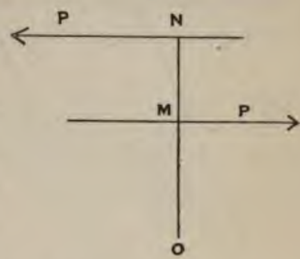


Thus, consider a capstan with six or eight bars radiating from the centre. If there are two men turning the capstan, working at the ends of two opposite bars, it is obviously immaterial which pair of bars they choose, so far as the turning effect on the capstan is concerned.

The turning effect of a couple is often technically termed a *torque*, and a couple of given *moment* produces a torque of definite amount, and this same torque could be produced by any other couple of equal moment.

These conclusions, which are suggested by experience, may be deduced from the laws of the composition of forces, as in the following propositions—

I. The moment of a couple about any point in the plane of the couple is constant.



Let the couple consist of two forces, each P, acting in lines whose distance apart is a.

Take any point O in the plane and draw OMN perpendicular to the lines of action of the forces.

$$\text{Moment of couple} = P.ON - P.OM,$$
$$= P(ON - OM),$$
$$= P \times MN = Pa,$$

which is independent of the position of O.

The distance MN is often termed the *arm* of the couple.

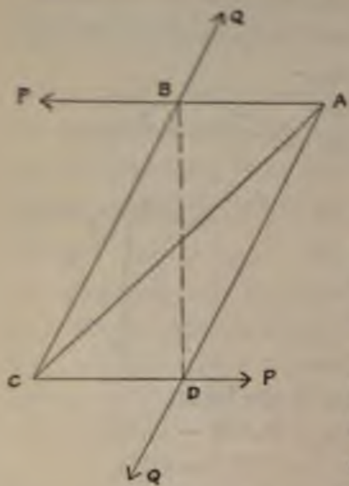
Example.

87. Draw the figure, and find the moment of the couple for the case when O is between the lines of action of the forces.

A couple produces a turning effect or *torque* which depends only on the plane of the couple and the moment of the couple, and *not* upon the magnitude or direction of the individual forces of the couple. In other words: (II.) Two couples in the same plane of equal moments, but

turning opposite ways, will balance one another if acting on a rigid body.

This proposition is an obvious deduction from Cor. III., p. 100, since the sum of the moments of the couples about any point in the plane vanishes. It may also be deduced from the parallelogram of forces, as follows—



Let P and P, Q and Q constitute the couples, their lines of action forming the parallelogram ABCD.

Take AB to represent the force P acting in the line AB, and then of course CD will represent the force P acting in CD.

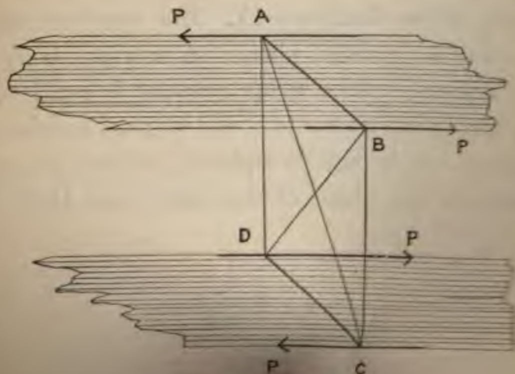
The moment of the P couple about D = moment of P in AB about D, which is represented by the area ABD, since AB represents P. But the moments of the couples are equal. Therefore the Δ ABD or the equal Δ CBD represents the mome[illegible] the Q couple, *i. e.* the moment of Q in CB ab[illegible] [illegible]efore CB represents Q on the same scale t[illegible] [illegible]nts P.

Theref[illegible] [illegible] of P in AB and [illegible] sented [illegible] [illegible]al AC of the [illegible] resp[illegible] [illegible]D and [illegible] di[illegible] the [illegible]

forces constituting the couples have as resultant two equal and opposite forces acting in the same line, *i. e.* are in equilibrium.

III. Two couples of equal moments in parallel planes, but turning in opposite senses, will be in equilibrium if applied to a rigid body.

Let the couple in one plane consist of forces P and P at a distance AB apart. The second couple, by the preceding proposition, is equivalent to a couple consisting also of two



forces P and P, [illegible] es of the former couple, and at a dista[illegible] l to AB. The [illegible] ABCD is the[illegible]

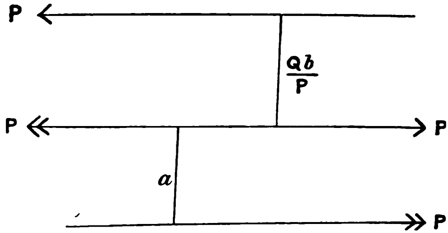
[illegible] ult[illegible] C is 2P at the [illegible] at B and P at D is 2P [illegible] point of BD. [illegible] lower one another. [illegible] and BD coincide [illegible] stituting the couples

have as resultant two equal and opposite forces acting in the same line, and are therefore in equilibrium.

This proposition serves to draw attention to the great importance of the words in the enunciation "applied to a rigid body." Two equal and opposite couples in parallel planes exert a powerful twisting effect on the body to which they are applied, but *assuming* that the body is rigid, they have, as the proposition shows, no tendency to turn it as a whole.

Resultant of two Couples.—IV. The resultant of two couples acting in parallel planes on a rigid body is a couple in a parallel plane whose moment is equal to the algebraic sum of the moments of the original couples.

The second couple may be replaced, without alteration, by a couple in the plane of the first (Prop. III.). Let



the first couple consist of two forces, each P, at a distance apart of a. The second couple, consisting of two forces, each Q, at a distance apart of b, can be replaced by a couple consisting of forces, each P, at a distance apart of $\frac{Qb}{P}$ (Prop. I.), and these forces can be turned round, without altering the moment of the couple, until they are

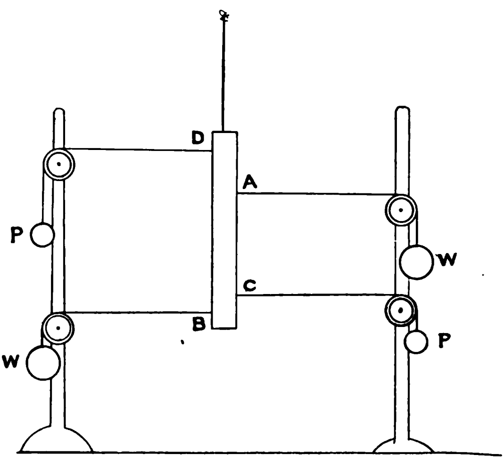
lel to the forces of the first couple (Prop. II.), and ier, until the lines of action of two oppositely directed s coincide. The figure is drawn for the case when couples are counter-clockwise in sense of rotation. now evident that the resultant is a couple consisting ro forces, each P, at a distance apart $a + \frac{Qb}{P}$, *i.e.* the ient of the resultant couple is $P\left(a + \frac{Qb}{P}\right)$, or $Pa + Qb$.

Example.

Draw the figure for the case when the second couple is wise.

Experiment 3.

) show that couples of equal moment but opposite



's balance if acting on a rigid body.

Using the frame and rod referred to at page 116, mark four points ABCD on the rod. Let AB = 5 inches, CD = 8 inches. Apply at A and B equal unlike forces, each 0·8 lb., by threads passing over pulleys as in the diagram.

Find what equal unlike forces applied at C and D will preserve equilibrium with the rod vertical. Find whether your result is altered if A and B are in front of the rod, and C and D at the back. For this purpose a rectangular block of wood, say 18 inches long by 4 inches square, is suitable.

EXAMPLES.

89. A picture weighing 14 lbs. hangs from a nail driven into the wall. What is the couple tending to bend the nail when the picture cord is close to the wall, say 0·05 inch from the wall, and what is the couple when the cord hangs at the head of the nail, 1·2 inches from the wall? In which position ought the cord to be placed?

90. Along the sides AB and CD of a square ABCD act forces each equal to 2 lbs. weight, whilst along the sides AD and CB act forces each equal to 5 lbs. weight. If the side of the square be 3 feet, find the moment of the couple that will give equilibrium.

91. When a boy sits on the branch of a tree the branch will probably break at the trunk if it breaks at all. Is it more likely to break if he sits near the trunk or further away? If the boy's weight is 100 lbs., what is the couple tending to break the branch when he sits 7 feet from the trunk?

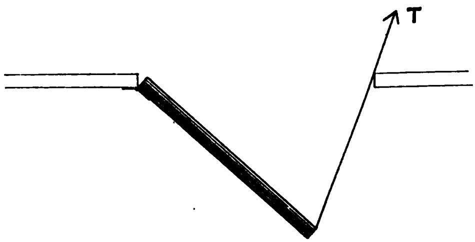
92. A beam 12 feet long and supported at the ends carries a load of 10 cwts. at a distance of 5 feet from one end. Neglecting the weight of the beam, find the thrust on each support and the couple that tends to bend the beam at the middle point. State the units in which your answers are given.

Moment of a Force about a Line.—Let the paper represent a horizontal plane. The moment of a force P in the plane about a point O in the plane is of course $P \times OM$. This

is also the moment of P about a line through O perpendicular to the plane. The moment of a force about any line at right angles to a plane containing the force is equal to the moment of the force about the point in which the line cuts the plane.

Examples.

93. A door 6 feet 6 inches by 3 feet 6 inches by $2\frac{1}{2}$ inches thick turns about a vertical axis. A force T of 5 lbs. is applied as in the



figure, which is a bird's-eye view or "plan" of the door and doorway. Find the moment of this force about the axis when the door is opened through an angle of 20°, 40°, 60°, 80°.

94. A round table of radius 3 feet and weight 80 lbs. has four legs at the corners of a square, each $2\frac{1}{2}$ feet from the centre. What is the least vertical force on the table which will lift two of the legs off the floor, and where must it be applied?

95. A uniform square plate of weight W rests horizontally on three props respectively at two adjacent angles and the middle point of the opposite sides. Find the greatest weight which can be placed at one of the other angles without causing it to topple over.

CHAPTER V

In this chapter some machines will be described in which the principles already described are turned to practical account.

Experiment 1.

Letter Weight.—As an experimental illustration of the principle of the lever we will show how to make a simple contrivance for weighing letters.

Take a piece of brass rod, say of section $\frac{1}{4}$ by $\frac{1}{8}$ inch, about 6 inches long. Clean it thoroughly and lacquer[1] it if desired. Quite close to one end drill a small hole, and through the hole pass a small ring. (A split ring is best if obtainable, if not a sufficiently good ring may be twisted up of copper wire.) Slip a bicycle trouser's clip through this ring. Hang a 4-oz. weight by a small piece of thread from the clip and balance the rod on a pencil placed close to the edge of the table. Take care that the pencil is at right angles to the rod, and find the position in which the rod just balances over the pencil. This is easily done by giving the pencil a slight rolling motion

[1] A good effect is produced with much less trouble by rubbing the brass irregularly with a bit of oil-stone, well oiled.

until the balancing position is found. Mark the spot immediately over the point of balance by a small ink line. File a fine nick on the upper side of the brass at this point.

Obtain in a similar way the marks for weights of 2 ozs. and $\frac{1}{2}$ oz. Write or scratch the numbers $\frac{1}{2}$, 2, and 4 by the respective marks. You may be able to etch them on in the chemical laboratory.

EXAMPLES.

1. Does the filing of the first small nick affect the accuracy with which the balance indicates a 4-oz. weight, a 2-oz. weight? Does the filing of the second mark cause any error in weighing 4 ozs., 2 ozs.? Is there any reason against finding the position of the mark for $\frac{1}{2}$ oz. first?

2. If it is wished to make the apparatus more compact, show that a weight may be fixed to one end of the rod.

3. Show that if the positions and distances apart of the 4-oz., 3-oz., and 2-oz. balancing points are given, the positions of the marks for any other balancing points can be calculated.

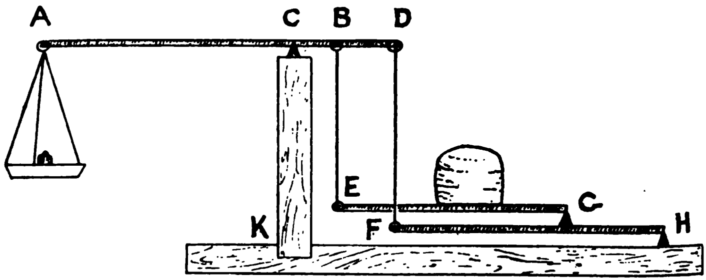
If the distance of 4-oz. from 3-oz. is x inches,
" " 3-oz. " 2-oz. is y inches,
show that distance of 2-oz. " 1-oz. is $\dfrac{y(x+y)}{3x-y}$ inches.

A weighing machine constructed on the above principle is termed a Danish steelyard.

Weighing-bridge.—Ordinary balances are seldom employed for weighing bodies heavier than 30 or 40 lbs. For weighing large heavy bodies, such as a loaded truck or an iron girder, a simple balance is quite unsuitable, as it would not only be of huge size and great cost, but to move the necessary counterbalancing weights, even sup-

posing that such large weights could be obtained, would be very inconvenient.

A weighing-bridge is a machine consisting of a combination of levers so arranged that the counterbalancing weight bears a certain fixed ratio to the weight of the load. Thus in the *decimal* weighing-bridge the counterbalance weight is only one-tenth of the weight of the load.



In the simpler forms of weighing-bridge there are three levers: the balance-arm AD, the platform EG, and the intermediate lever FH. The load is placed on the platform and it is counterbalanced by weights in a scale-pan suspended from the end A of the balance-arm. The platform EG and the lever FH are connected with the balance-arm by rods BE and FD, resting on knife-edges at their extremities so as to give freely moving joints, concentrating the pulls in these rods at definite points of the balance-arm. When the balance is in equilibrium the three levers are horizontal and the two rods vertical.

The lever FH moves about a fixed knife-edge at H as a fulcrum, while the platform has a knife-edge at G, resting

on the lever FH. The balance-arm turns about a knife-edge at C on the end of the upright CK.

Suppose that there is equilibrium when there is no weight in the scale-pan and no load on the platform.[1] Now let a body of weight W be placed on the platform and counterbalanced by a weight w in the scale-pan. Since the platform is supported at E and G, the effect of the load is to produce a pull in the rod EB, which we may denote by P, and a pressure on the knife-edge G, which we may denote by Q, and the sum of these must be equal to W (Why ?).

Now the pressure Q acting on the lever FH, which moves about H as a fulcrum, will produce a pull in the rod FD of amount $Q \times \frac{GH}{FH}$ (Why ?). The effect on the balance-arm produced by the load is thus (1) a pull in the rod BE of amount P, and (2) a pull in the rod DF of amount $Q \times \frac{GH}{FH}$. The moments of these two pulls about the fulcrum C must balance the moment of w, and we therefore have the equation—

$$w \times AC = P \times CB + Q \times \frac{GH}{FH} \times CD.$$

Now let us suppose that B divides CD in the same ratio that G divides HF, *i.e.* that $\frac{GH}{FH} = \frac{CB}{CD}$,

or $$CB = CD \times \frac{GH}{FH},$$

[1] If this is not the case, weights must be put on the platform or in the scale-pan till there is a good balance.

the equation then becomes—

$$
\begin{aligned}
w\text{AC} &= \text{P} \times \text{CB} + \text{Q} \times \text{CB}, \\
&= (\text{P} + \text{Q})\text{CB}, \\
&= \text{W} \,.\, \text{CB}.
\end{aligned}
$$

If then the balance is so constructed that B divides CD in the same ratio as G divides HF, we may regard the load W as suspended directly from B. If now we wish to have a *decimal* balance we must make BC one-tenth of AC, if a *centesimal* balance BC must be one-hundredth of AC.

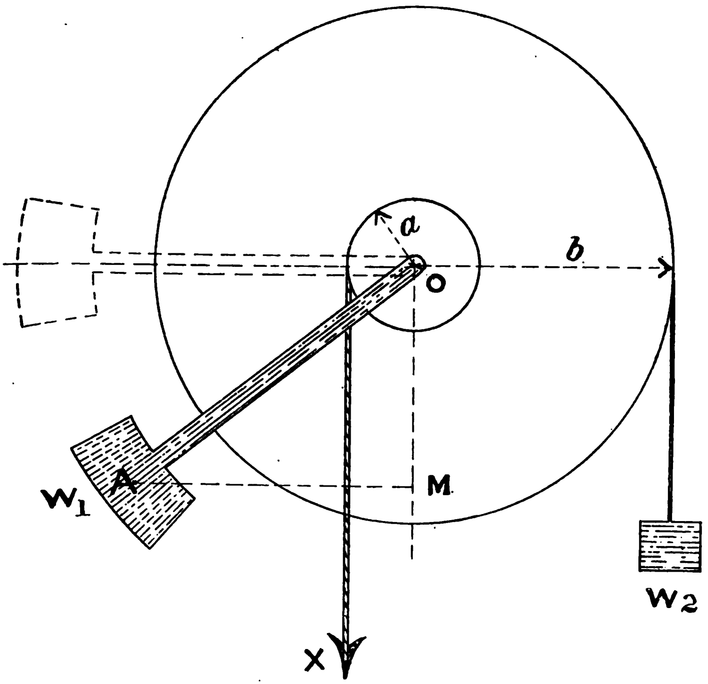
Nothing has been said about the weights of the different parts of the bridge. There will, of course, be tensions in BE and DF initially due to the weights of the lever and platform, but we may leave these entirely out of account if, before commencing the weighing, the balance is in equilibrium. The pulls produced by the load will be in addition to the initial pulls, and the equations we have put down will still hold.

Will the position of the load on the platform affect the result in any way?

A simple model weighing-bridge may be easily made, filing up knife-edges of brass and using fine chain for the suspension rods.

An instrument of this kind is employed for rapidly counting out a large number of small similar objects such as screws. If a decimal balance were employed a single screw on the one side would balance ten screws on the other. These ten transferred to the other side would balance one hundred, and so by these means any desired number can be rapidly counted out.

"**Penny-in-the-slot" Machine.**—The ordinary "penny-in-the-slot" weighing-machine is a somewhat complicated piece of mechanism, owing to the necessity of setting the mechanism at work with a penny, and of preventing more than one person being weighed for the same penny. But



the mechanical principle on which the actual weighing depends is simple. The machine may, in this aspect, be considered as a "wheel and axle."

A circular disc can turn about an axis at right angles to its plane through its centre O. A weight W_2 hangs by a

flexible cord, which is coiled on the circumference. The person to be weighed stands on the platform and (as in the weighing-bridge just described) produces an additional pull proportional to his weight in a chain coiling on a concentric groove of radius a. A weight W_1 is fixed to an arm projecting from and rigidly attached to the disc.

When there is equilibrium the sum of the moments about O of W_1 and of the pull of the chain must equal the moment about O of the weight W_2. But the moment of W_2 about O is the same for all positions of the disc. Hence the diminution in the moment of W_1, due to the disc turning through any angle θ, must just equal the increase in the moment of the pull in the chain due to the additional load on the platform. Initially, before the person steps on the platform, let the arm AO (r) be horizontal.

The moment of W_1 about O was originally $W_1 r$. When the disc has turned through an angle θ it is $W_1 \times$ AM, or $W_1 r \cos\theta$. Hence the decrease in the moment of W_1 is $W_1 r(1-\cos\theta)$.

The additional pull in the chain being X, the increase in the moment is Xa, and therefore—

$$Xa = W_1 r(1 - \cos\theta),$$

or
$$\cos\theta = \frac{W_1 r - Xa}{W_1 r} = 1 - \frac{Xa}{W_1 r}.$$

For each given weight on the platform there is a given additional pull X on the chain, and therefore a certain definite inclination θ of the arm, determined by the above equation.

EXAMPLE.

4. Given $W_1 = 7$ lbs., $r = 12$ inches, $a = 1\frac{1}{2}$ inches. Find cos θ for weights of 3, 4 . . . 15 stones, supposing that the extra pull in the chain is one-tenth of the actual load on the platform.

Draw a graph showing the relation between X and cos θ. Hence draw also a graph connecting X and θ.

In practice, the angular movement of the disc is magnified, so to speak, by a set of cog-wheels, which, when a given load is on, bring a projecting pin into a definite position on the circumference of the dial of the instrument. The penny actuates a pointer, which if it were not stopped by the pin would move right round the circumference of the dial. It is, however, stopped by the pin, and indicates the weight corresponding to that position of the pin.

The Principle of Work.—Much ingenuity has been expended in trying to make a machine in which a weight, by its descent through a given distance, will raise an equal weight through a greater distance.

All such attempts have failed, and the conviction has gradually grown stronger that this failure is not due to a lack of skill or ingenuity on the part of the artist, but to the impossibility of the task.

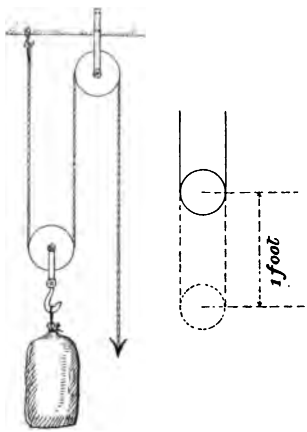
There are certain stores of "energy" or power of doing work in the universe, and mankind has learned to utilize some of these stores. But, as the student's knowledge of the forms of energy widens, he will only realize more fully the truth of the statement that man cannot create energy but can only use it.

We shall state this principle in the very limited form *which our present* experiments will suggest. A weight

of W lbs. descending through h feet cannot perform more than Wh foot-pounds of work, and to raise a weight or push back a resistance of W lbs. through a distance of h feet involves the doing of at least Wh foot-pounds of work. "That which can raise a weight through a certain height can raise n times the weight through $\frac{1}{n}$ of the height." The principle of work was enunciated in these terms in the thirteenth century, by Jordanus, of Nemi (*Duhem*, p. 358).

EXAMPLES.

5. The diagram shows an arrangement of pulleys for hoisting up weights such as a sack of corn into a warehouse. How many feet of rope must the man pull in order to raise the load 1 foot? (see figure). If the load weighs 140 lbs., what is the least force which will enable the man to lift the weight.



[The ratio $\frac{\text{load raised}}{\text{force applied}}$ is termed the *mechanical advantage* of a machine.]

6. A man winds up a bucket of water weighing 60 lbs. through a height of 50 feet by a wheel and axle. The rope coils on the axle, whose diameter is 9 inches, and the radius of the wheel is 20 inches. What is the least force applied tangentially to the wheel which will raise the bucket? How many foot-pounds of work at least must the man do? If one-third of the work done by the man is wasted, owing to the stiffness of the mechanism, how many foot-pounds of work must the man do?

In practice some of the work expended in working a

machine is always wasted, or lost owing to friction of bearings, stiffness of ropes, and similar causes.

The fraction $\dfrac{\text{useful work realized}}{\text{total energy expended}}$ is termed the *efficiency* of the machine. In a perfect machine the efficiency would be unity.

Example.

7. If a weight of 1 ton is to be raised 20 feet by a machine whose efficiency is 45 per cent., or 0·45, what is the total energy expended, and how much work is wasted?

In a "block" (see figure) a number of pulleys run (each separately from the rest) on an axle. A tackle consisting of a pair of blocks is much used for shifting heavy guns, and similar purposes.

A is the "standing" block, B is the "running" block. A rope, fastened to B, goes round first pulley at A, round first pulley at B, round second pulley at A, round second pulley at B, round third pulley at A, round third pulley at B, and then to the hand of the man working the tackle.

We have already seen that, if all the pulleys work

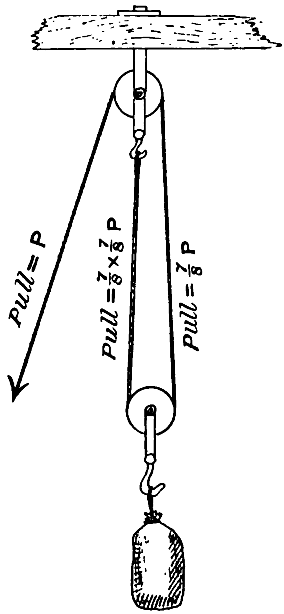
perfectly smoothly the tension of this rope will be the same throughout.

EXAMPLES.

8. What is, on this supposition, the total pull on the block B due to a pull of 5 cwts. in the rope? What amount of rope must be pulled out in order to move the block B 1 foot? What is the ratio of the distance moved by the pulling force to the distance moved by the resistance? This ratio is called the *velocity ratio* of the tackle.

9. Compare the velocity ratio of the previous tackle with what it would be if A were the running, B the standing block.

10. Suppose the efficiency of the tackle in Example 8 to be 50 per cent., what pull is required to overcome a resistance of a ton?



In order to make a reasonable allowance for the loss of efficiency due to the stiffness of the ropes, the want of freedom in the running of the pulley and similar causes, it is often assumed[1] that a rope which is being pulled on loses a certain fraction of its tension each time it passes over a pulley.

We will calculate the efficiency of a tackle (see figure) on the assumption that the fraction lost is $\frac{1}{8}$.

Considering the lower block we have—

$$\text{Lifting force on it} = \tfrac{7}{8}\text{P} + \tfrac{7}{8} \times \tfrac{7}{8}\text{ P}.$$

$$\text{Load on it} = \text{W, say.}$$

[1] This assumption has some theoretical justification.

The weight of the lower block is supposed included in W.

If the lifting force just raises the load, we have—

$$\tfrac{7}{8}P + (\tfrac{7}{8})^2 P = W,$$

or

$$P = \frac{64}{105} W.$$

From the construction of the tackle we see that to raise the load 1 foot we must pull in 2 feet of rope.

$$\left.\begin{array}{l}\text{Hence useful work realized} = \\ \text{work done in raising W 1 foot}\end{array}\right\} = W \times 1,$$

$$\left.\begin{array}{l}\text{total energy expended} = \text{work} \\ \text{done in exerting a pull P} \\ \text{through 2 feet}\end{array}\right\} = P \times 2 = \frac{64}{105} W \times 2,$$

$$\therefore \text{efficiency of tackle} = \frac{W}{\frac{128}{105}W} = \frac{105}{128},$$

$$= 0{\cdot}82.$$

so that 82 per cent. of the work expended is productive and 18 per cent. is unproductive, or wasted.

EXAMPLES.

11. Assuming that when a rope passes round a pulley its tension falls off in the proportion of $\mu : 1$, so that, if T was the original pull then μT is the pull after passing *once* round a pulley, $\mu^2 T$ after passing twice, and so on; calculate the efficiency of the tackles of Examples 8 and 9. Give the numerical results for $\mu = 0{\cdot}5$, $\mu = 0{\cdot}8$.

12. A wheel and axle, efficiency 75 per cent., is employed to pull in rope from a block and tackle, efficiency also 75 per cent. What is the efficiency of the combination?

13. In a lifting tackle the upper fixed block has three sheaves and the lower movable one two sheaves. A pull of 80 lbs. is required to lift 3 *cwts. Find the efficiency* of the tackle and the amount of work *lost in friction* when the load rises 6 inches.

14. In a machine for raising weights, a pull of 50 lbs. will raise a load of 2 cwts., but 6 feet of rope must be pulled in to raise the load 1 foot. What is the efficiency?

15. When one of the keys of a pianoforte is depressed through $\frac{3}{5}$ inch the hammer is raised through a vertical height of 2 inches. A force of 2 oz. weight will just depress the key. Find the weight of the hammer. Neglect friction and all weights except that of the hammer. [Army]

Weston's Differential Pulley, shown in figure 1, is a machine used for lifting heavy weights. By its use one man can raise a weight of a ton or more. It consists of a block OAB, which can turn round an axle at O. On the rim of the block are two circular grooves—one of slightly greater radius than the other. This axle is fixed at a suitable height above the ground. An "endless" chain passes, as shown in the figure (2), round the larger groove, round a pulley C, from which hangs the weight which is to be raised, then round the smaller groove, and so back to the larger groove. It cannot slip on the grooves, as ridges are cut to prevent this.

Let the radius of the larger groove be 6 inches, and of the smaller groove 5 inches.

Let the workman exert a pull P, and pull 12 inches of chain off the larger groove at B. This causes 12 inches of chain to coil up on the larger groove at A, and $\frac{5}{6} \times 12$ inches, or 10 inches, of chain to uncoil from the smaller groove at D.

The result then is that the length of chain between A and C and D is decreased by $\frac{1}{6} \times 12$ inches, or 2 inches.

Now, if the length of chain ACD diminishes by 2 inches,

Fig. 1.

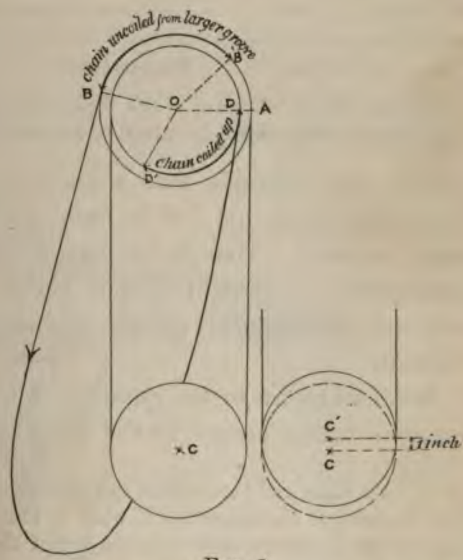


Fig. 2.

[The block turns through angle BOB′ = DOD′ as 12 inches of chain are pulled off at B.

B′ comes to B.

D′ ,, D.]

the pulley C must rise to C′, where CC′ = 1 inch (see figure 2).

Now, P being the pull exerted by the workman in pounds—

$$\text{Work done by P} = 12\text{P inch-pounds},$$
$$\text{work utilized in lifting W} = 1 \times \text{W inch-pounds}.$$

Hence if all the work done by P is realized in lifting the weight we should have—

$$12\text{P} = \text{W},$$

showing that a weight of 12 cwts. could be raised by a pull of 1 cwt.

Examples.

16. Suppose the efficiency of the Weston pulley were 80 per cent., what load could be raised by a pull of 1 cwt.?

It is very desirable that a machine used for raising heavy weights should not be capable of "*taking charge*" as it is called. That is to say, if the lifting force is inadequate, the weight should simply remain at rest and not take charge of the proceedings and descend violently.

We shall return to the question of how to prevent the machine taking charge in the chapter on friction.

17. In a block and tackle with two sheaves in each block and the rope fastened to the upper block, what is the velocity ratio, *i. e.* the ratio of the distance traversed by the end of the rope to the distance traversed by the load?

18. It is found that with this block and tackle that a pull of 40 lbs. is required to raise a load of 56 lbs. What is the efficiency of the machine for this load, *i. e.* the ratio of the work given out by the machine to the work put in?

And if a pull of 65 lbs. will raise a load of 112 lbs., what is the efficiency for this load?

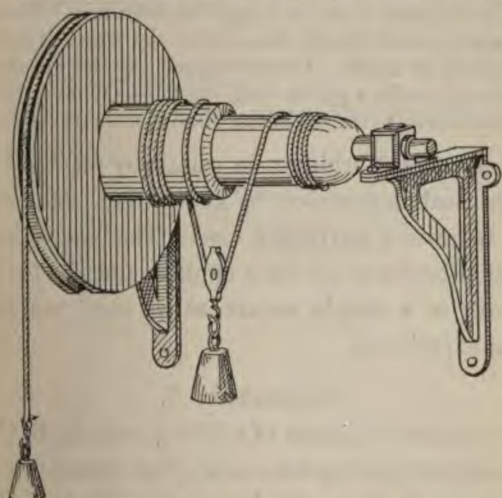
19. An ordinary winch is used to lift water out of a well. It is found that a load of 112 lbs. hanging on the rope can just be balanced by a force of 30 lbs. acting at the handle in a direction tangential to its circular path. The length of the handle is 20 inches, and the diameter of the barrel of the winch is 8 inches. Calculate the efficiency of the winch. [Army]

20. Find the mechanical advantage of a system composed of a fixed and a movable pulley with three ropes at the lower or movable block, assuming that, owing to friction, and the stiffness of the rope, when a rope passes round a pulley the tensions on either side of the pulley are in the ratio 7 : 8.

Define the "efficiency" of a machine, and find the efficiency of the above tackle, on the assumptions stated.

21. Explain the "principle of work," as applied to machines. Apply it to find the relation between the power and weight on an inclined plane, where the power acts parallel to the plane. [Army]

The Chinese Wheel and Axle (sometimes called the



"Differential axle").—The barrel is in two portions of different radii, r_1 and r_2 (see figure).

The rope is endless, that is, it coils up on one part of the barrel as it uncoils from the other.

EXAMPLES.

22. If $r_1 = 8$ inches, $r_2 = 6$ inches, find what weight can be raised by a force of 1 cwt. applied at right angles to a handle 18 inches long, and how many turns of the handle are required to raise the weight 5 feet, neglecting friction.

23. In a wheel and differential axle the diameter of the wheel is 30 inches, and the diameter of the two parts of the axle are 10 and 12 inches. If the efficiency of the machine is 45 per cent., find the tangential pull on the wheel necessary to raise a load of 500 lbs.

24. Draw carefully a diagram of that system of pulleys in which a continuous rope passes round all the pulleys, showing three portions of rope at the lower block.

Suppose a mark on each of these three portions, the marks being originally in a horizontal line. Draw a second diagram showing the positions of the marks after the weight has been raised 1 foot.

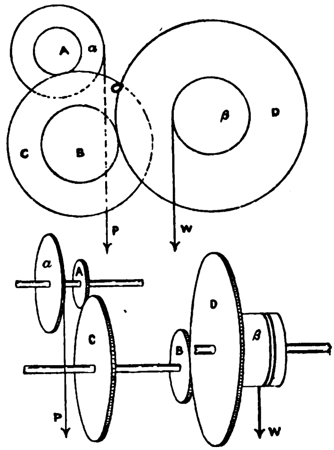
25. In a wheel and axle the diameter of the wheel is 3 feet, and that of the axle 18 inches. A rope wrapped opposite ways round the wheel and axle carries a pulley weighing 8 lbs. What force must be applied tangentially to the rim of the wheel to maintain equilibrium?

Efficiency of a Machine.—In using any machine it is of great practical importance to know what force must be applied to raise a particular load. The law which gives the relation between the force applied and the load raised is generally of a simple nature, as we shall see from the following experiment.

EXPERIMENT 1.

In the annexed diagram of a lifting crab A, B, C and D are spur-wheels gearing into each other by means of teeth. On the same shaft as the large wheel D is a barrel β, round which is wound a strong rope to support the load *W*. On the same axle as the small wheel A is a disc *a*

with a groove cut in the edge, in which lies the cord which supports a scale-pan on which is placed the weight which is to serve as the lifting force. In practice *a* would be replaced by a handle worked by manual labour, but for purposes of experiment it is more convenient to have a disc with a groove, and to use weights. In an actual experiment we determined the smallest weight in the scale-pan which would just raise a load of 50 lbs. In a similar way the weights in the scale-pan required to raise 75, 100, 125 and 150 lbs. were found.



Load.	Weight in scale-pan (including the weight of scale-pan itself).
W	P
50 lbs.	1·0 lbs.
75 „	1·4 „
100 „	1·8 „
125 „	2·2 „
150 „	2·6 „

Plotting these values on squared paper, we find that the five points lie on a straight line. [The values of P

were only taken to the nearest tenth of a pound, and if smaller weights had been available, so that the values of P could have been more accurately found, we should have found that the points lay approximately on a straight line. We should then have drawn the straight line of "closest fit," that is, the straight line which coincides most nearly with the points. The position of this line is most easily obtained by stretching a piece of cotton across the paper.]

Since the values of W are so much greater than those of P, it is as well to use a much larger scale for P than for W. In the diagram actually drawn the scale for P was thirty times the scale for W.

The relation between P and W may now be expressed by the equation—

$$P = aW + b.$$

To get a and b take two pairs of values of W and P, represented by two points some distance apart, on the line, say (50, 1) and (100, 1·8).

$$\begin{aligned} 1 &= 50a + b, \\ 1{\cdot}8 &= 100a + b, \\ 2 &= 100a + 2b, \\ \therefore b &= 0{\cdot}2, \end{aligned}$$

and $$a = \frac{0{\cdot}8}{50} = 0{\cdot}016,$$

$$\therefore P = 0{\cdot}016\ W + 0{\cdot}2.$$

If we produce the line representing the relation between P and W backwards, we find that it cuts the P axis at the point + 0·2. A force then of 0·2 lb. should work the

machine unloaded. This was found to be approximately true, for the scale-pan itself weighed 0·2 lb., and was just sufficient to put the machine in motion when unloaded.

We arrive at the following result, which is very approximately true for many machines. The applied force requisite to raise a given load is the sum of the applied force necessary to set the machine in motion when unloaded, and a definite fraction of the load.

Form a third column, showing for each value of W, the corresponding value of $\frac{W}{P}$.

W	P	$\frac{W}{P}$
50	1·0	50
75	1·4	53·5
100	1·8	55·5
125	2·2	56·8
150	2·6	57·6

Summary.—We will repeat the three important definitions[1] which have been given. In any machine—

(1) *Efficiency*

$$= \frac{\text{useful work realized}}{\text{total energy expended}}.$$

[1] The terms "efficiency" and "velocity ratio" are always used in the sense here defined. Some writers, however, have defined the "mechanical advantage" as the ratio $\frac{\text{load}}{\text{applied force}}$ when the efficiency is perfect, leaving the last five words to be understood.

L

(2) *Velocity ratio*

$$= \frac{\text{distance moved by the applied force}}{\text{distance the load is moved}}.$$

(3) *Mechanical advantage*

$$= \frac{\text{load moved}}{\text{force applied}}.$$

Notice that efficiency

$$= \frac{\text{useful work realized}}{\text{total energy expended}},$$

$$= \frac{\text{load} \times \text{distance load moves}}{\text{applied force} \times \text{distance through which applied force acts}},$$

$$= \text{mechanical advantage} \times \frac{1}{\text{velocity ratio}},$$

$$= \frac{\text{mechanical advantage}}{\text{velocity ratio}}.$$

It is not sufficient that we should be able by the aid of a machine to lift a load which we could not lift directly, it is also of great importance that the machine should work as economically as possible. That is to say, as much as possible of the work or labour put into the machine ought to be usefully employed in raising the load. Hence the importance of knowing the efficiency.

The Lifting Crab.—If a cog-wheel with m teeth gears with one of n teeth, then if the first makes n complete turns the second will make m complete turns.

Hence, in the case of the lifting crab the velocity ratio can be calculated by counting the teeth on each spur wheel, and measuring the diameter or the circumference of the barrel β, and also of the disc a.

Velocity ratio

$$= \frac{\text{circumference of } a}{\text{circumference of } \beta} \times \frac{\text{teeth of C}}{\text{teeth of A}} \times \frac{\text{teeth of D}}{\text{teeth of B}}$$

$$= \frac{22{\cdot}7}{6{\cdot}5} \times \frac{66}{22} \times \frac{112}{16} \text{ in the machine experimented on,}$$

$$\therefore = \frac{22{\cdot}7}{6{\cdot}5} \times 3 \times 7 = 73{\cdot}4.$$

In many machines the velocity ratio is more easily found experimentally by finding how far the load is raised when the applied force descends through a given distance.

The *velocity ratio* for any machine is quite independent of the load. The *mechanical advantage* and the *efficiency* will, however, vary for different loads.

In the case of the lifting crab, in which we found that $P = 0{\cdot}016W + 0{\cdot}2$,

$$\text{mechanical advantage} = \frac{W}{P} = \frac{W}{0{\cdot}016W + 0{\cdot}2} = \frac{1}{0{\cdot}016 + \frac{0{\cdot}2}{W}}$$

As the load increases, $\frac{W}{P}$ increases towards the value $\frac{1}{0{\cdot}016} = 62{\cdot}5$. The mechanical advantage therefore increases with the load, and approaches a definite limiting value.

$$\text{Efficiency} = \frac{\text{mechanical advantage}}{\text{velocity ratio}},$$

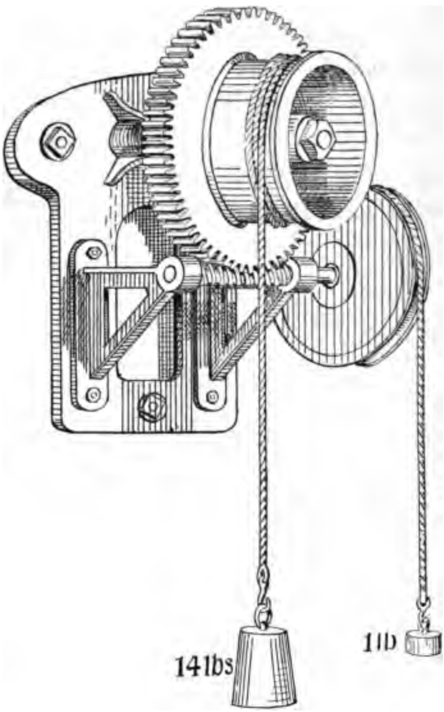
$$= \frac{1}{0{\cdot}016 + \frac{0{\cdot}2}{W}} \times \frac{1}{73{\cdot}4},$$

$$= \frac{1}{1{\cdot}17 + \frac{14{\cdot}68}{W}}.$$

The efficiency, therefore, also increases with the load, and tends towards the value $\frac{1}{1 \cdot 17} = 0 \cdot 855 = 85\frac{1}{2}$ per cent. when the load is very great.

EXAMPLES.

26. Work out the efficiency from the table on p. 145, and state its value for each of the loads.



27. In a crab, the axle which is operated by the turning handle carries a wheel with 13 teeth, this wheel engages with a large wheel with 79 teeth, which is keyed on to the axle which rotates the winding barrel. Diameter of barrel, 12 inches; leverage of turning handle, 18 inches. Find the efficiency of the crab if a force of 20 lbs. applied to the handle will just raise a weight of 290 lbs.

EXPERIMENT 2 for determining the efficiency of a machine.

A convenient machine for this experiment is a worm and wheel gear,[1] and the following

[1] Supplied by Messrs. Cusson, of Manchester.

directions are adapted to this, but the experiment may be carried out with pulleys or cog-wheels, or with a bicycle if the latter can be conveniently slung from the roof. A worm or screw passes through fixed bearings, and has on it a disc, in the rim of which is a groove. From a cord coiled on this groove weights can be hung, causing the disc, and with it the worm, to turn. The worm actuates the "worm wheel." On a drum or groove, forming part of the worm wheel, is coiled a rope, from which the load[1] hangs.

Worm and Wheel.—The following values were obtained in an actual experiment with a worm and wheel—

P (ozs.)	8	13	18	25	31	38	44	51	63	76
W (lbs.)	7	14	21	28	35	42	49	56	70	84

Example.

28. Show that the graph of P and W is very nearly a straight line passing through the origin. Draw the straight line of "closest fit" which passes through the origin, and hence determine the mean value of the mechanical advantage. What inferences do you draw from the fact that the straight line passes through the origin?

When the applied force moved through 3 feet 4 inches the load was raised 0·55 inch. Find the mean value of (1) the mechanical advantage, (2) the efficiency.

Suspend weights, W, of 5, 10, 15, 20 and 25 lbs. from the wheel, and find what weights, w, will *just* raise them by means of the worm. This may be done with sufficient accuracy by the use of a spring balance, provided care is taken to maintain a slow, steady movement.

[1] In experiments such as these, in which weights of considerable magnitude are used, care should be taken that if anything gives way the load can only fall a few inches, and not on any one's foot.

Construct a graph showing the relation between W and w. Measure the vertical height which W rises when w descends 4 feet. Complete the following table—

W.	w.	Work done by descent of w.	Work done in raising of W.	Difference, viz. work lost.	Efficiency.

Plot the graph, showing the relation between the efficiency and the load.

Examples.

29. In Weston's blocks, suppose 17 links of the chain would fit round one sheave of the upper block, 16 links round the other, and 13 links round the sheave of the lower block. What is the velocity ratio?

If 40 per cent. of the applied force is effective, what is the mechanical advantage?

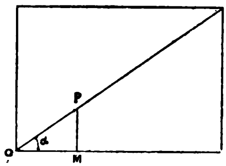
If the percentage efficiency was the same for the reverse motion and all weights but the load were negligible, would the load run down?

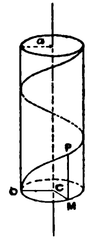
30. The pedals of a bicycle, instead of being fixed on the axle of the driving wheel, are fixed on another axle which carries a wheel with 20 cogs. The driving wheel has on its axle a wheel with 9 cogs, and these two cog-wheels are connected by a chain. What is the velocity ratio? And why is this arrangement adopted?

The Screw.—Every one is familiar with the appearance of a screw. An ideal screw may be supposed to be formed thus. Take a rectangular piece of paper, draw a diagonal, and fold the paper on to the surface of a cylinder.

The diagonal forms a screw line on the surface of the

cylinder. Let the diagonal of the rectangle make an angle



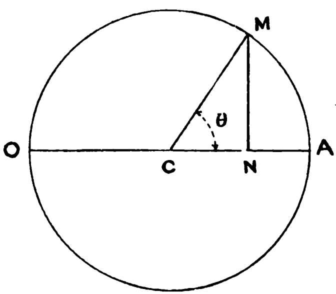


a with the base. Then, comparing the above figures, in which corresponding points are marked by the same letters, we see that if P is a point on the screw thread, PM the perpendicular on the base, and $\angle$ OCM $=\theta$, then—

$$\text{height of P} = \text{OM} \tan a,$$
$$= a\theta \tan a,$$

so that the position of P may be defined by the two quantities θ and $a\theta \tan a$.

A cylinder of 4 cm. radius is 10 cm. high. Draw a rectangle which would just fold on to the cylinder so as to cover the curved surface. Rule lines on the rectangle which will fold into a screw line making five complete turns.



The rectangle is called the "development" of the cylinder.

The distance between two consecutive threads measured

parallel to the axis is generally termed the "pitch"[1] of the screw and denoted by p.

An actual screw cannot, of course, be a geometrical line. It may be thought of as a surface containing a series of screw threads.

Experiment 3.

Take a cylinder, wrap a sheet of paper on it, and fasten with elastic bands. Measure the circumference of the cylinder after the paper is on. This may be done by wrapping several turns of fine cotton round the cylinder, counting the number of turns, and measuring the length of cotton used. Now draw a rectangle whose base is equal in length to the circumference of the cylinder, and height equal to the height of the cylinder. Draw a line on this rectangle which, when the rectangle is folded on to the cylinder, will make a screw of 1 inch pitch.

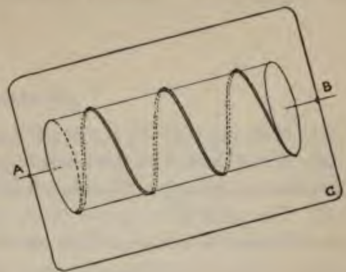
Experiment 4.

The Screw of Archimedes.—This device is said to have been discovered by Archimedes. It consists (see figure below) of a hollow pipe wound in a screw on a cylinder. The axis AB is inclined to the horizontal. If the cylinder is made to revolve small objects put in at the lower end of the pipe will be conveyed to the upper end.

The instrument was used in the old days for raising cannon-balls from a quay to a ship. It is still used

[1] The quantity $a \tan \alpha$ is sometimes called the "pitch" of the screw thread (Routh, *Statics*, p. 210), but the more usual meaning is that stated in the text.

occasionally for raising liquids or wheat. A small object, *e.g.* a shot, is put in at the lower end. Where is it after one complete turn?



Construct a screw of Archimedes with india-rubber tubing fixed to a wooden cylinder. What is the ratio of the greatest height an object can be lifted thereby to the pitch of the screw?

The Screw as a Measuring Instrument.—A screw fits in a nut. Suppose the nut held fixed and the screw turned. It moves forward in the nut. The distance the screw advances is proportional to the angle turned through, for in the figure (p. 151) PM is directly proportional to OM, *i.e.* to the angle OCM.

Now, it would be very difficult to make a good scale to read $\frac{1}{1000}$ inch, and we should require a microscope to use it. But suppose a screw is made so that its point moves $\frac{1}{10}$ inch from the nut when the screw makes one complete turn, in other words, suppose the pitch of the screw is $\frac{1}{10}$ inch, then $\frac{1}{100}$ of a turn feeds the point forward $\frac{1}{1000}$ inch.

Evidently, if we can measure the angle through which a screw turns, and we know its pitch, we can readily calculate how much it moves forward when turned through any given angle.

For a description of the micrometer gauge, spherometer,

and other instruments to which this principle is applied, see books on "Practical Mathematics."

EXAMPLES.

31. How far will the screw advance when it is turned through 1°?

32. What should be the pitch of a screw if turning it through 10° advances it through a fixed nut 0·001 inch?

33. The pitch of a screw is $\frac{1}{40}$ inch, and its head is graduated into 25 equal divisions. How much will the screw advance through a fixed nut if it is turned through one division?

The Screw as a Means of exerting Pressure.—A smooth screw working in a fixed nut will advance through the nut if it is turned.

Suppose a force P applied to turn the screw by means of a handle of length l, as in a letter-copying press.

Let p be the pitch of the screw, that is, the distance it advances in one complete turn.

Let the point of the screw encounter a resistance X. Then we have—

∴ Work done by P in making one complete revolution $\Big\} = 2\pi l\text{P}$.

Work done in thrusting the resistance X back through a distance p $\Big\} = \text{X}p$.

If therefore the efficiency is perfect—

$$X = P \times \frac{2\pi l}{p}.$$

Examples.

34. In a copying-press the length of the handle is 8 inches, and a complete turn of the handle causes the screw to advance $\frac{1}{20}$ inch. Find the pressure created by a force of 5 lbs. applied tangentially to the handle, neglecting friction.

35. If the efficiency of the machine in Example 34 is 40 per cent., what is the thrust exerted?

36. Power is applied to a screw-jack at the end of a horizontal arm 18 inches long (measured from the axis). The screw has a pitch of 1 inch. If there were no frictional losses what pull would have to be exerted at the end of the arm in order to lift half-a-ton? What would be the pull if only 35 per cent. of the work done on the arm is available for lifting purposes?

37. A lifting jack has a screw with two threads to an inch and the lever arm is 20 inches long. If 70 per cent. of the work expended is lost in friction, what force at the end of the lever arm is required to raise 4 cwts.?

38. The efficiency of a screw-jack is 40 per cent. The ratio of the power to the weight is $\frac{1}{200}$. Find the pitch if the lever is 18 inches long.

39. In operating the lever of a lifting jack it is assumed that a man can exert a horizontal force of 100 lbs. The pitch of the screw is $\frac{3}{8}$ inch and the efficiency of the machine is 30 per cent. Calculate the minimum length of the lever which is capable of operating the jack when the load to be raised is 4 tons.

40. By the principle of work find the relation between the power and the weight in the case of an ordinary screw-jack in which the screw is of $\frac{1}{2}$-inch pitch and the radius at which the power is applied is 2 feet.

41. A weight is lifted in a screw-jack, pitch $\frac{1}{4}$ inch, the force being applied at right angles to a lever 30 inches long. The load in tons and the force in pounds are as follows—

Load	1	2·5	5	7	8	10
Force	12	16	23	28	32	36

Find a formula connecting the two, and calculate the efficiency for the 10-ton and 1-ton loads.

42. A certain machine is so constructed that a load of W lbs. can just be lifted 1 foot by a force of P lbs. moving through a distance of 30 feet, where $P = 3·7 + 0·058\ W$. Find the efficiency of the machine when W has the values 10, 100, 1000 and 10,000 in succession.

[Army]

43. In an experiment with a screw-jack the following results were obtained—

Load W lifted in pounds.	Power P applied in pounds.
100	12·6
120	13·8
140	15·7
160	17·6
180	19·6
200	21·5

Plot these observations. Draw a fair line through them, and from your diagram read off the values of a and b which satisfy the equation—

$$W = a + bP.$$ [Army]

44. The driving-wheel of a motor-car is 3 feet in diameter, and a wheel 20 inches in diameter keyed to its axle is connected by a belt to a wheel 4 inches in diameter. The couple applied to this last wheel being G (in pounds-feet), what is the force urging the car forward?

[Hint: The work done by the couple in turning the wheel once is $2\pi G$ foot-lbs. Prove this.]

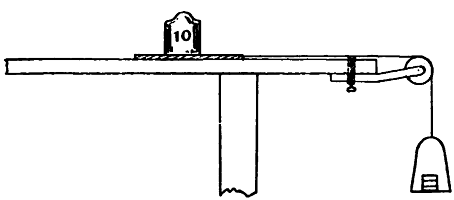
CHAPTER VI

FRICTION

PLACE a book on the table and push it gently. It will not at first move. Yet it is acted on by a horizontal force, the pressure of the finger; and this force, however small it were, would move the book if it were the only horizontal force acting. We see that the resistance to motion is due to an opposing horizontal force, called a force of friction, between the book and the table. We proceed to experiment somewhat more carefully.

EXPERIMENT 1.

A small pulley is screwed to the edge of a board. A fine thread passing over this pulley has a hook at one end,



and a light scale-pan at the other. A thin slab of wood or metal is placed on the board, a weight of say 10 lbs.

placed on the slab, the string hooked to an eye in the slab and a weight of $\frac{1}{2}$ lb. placed on the scale-pan. The slab does not move. What force of friction is acting?

[The string should be horizontal and the slab should be thin.]

Cut the string. If the force of friction continued to act it would be the only horizontal force on the slab, and consequently the slab would move to the *left*. As it does not do so, we infer that the force of friction ceases to act as soon as the force which tends to move the slab ceases.

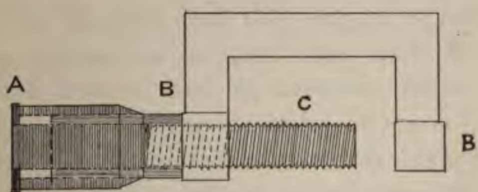
Place another $\frac{1}{2}$-lb. weight on the scale-pan. If the slab still does not move, what is now the force of friction? As the pulling force can be altered without motion taking place, we infer that the force of friction adjusts itself automatically to just the amount requisite to prevent motion; being $\frac{1}{2}$ lb. when $\frac{1}{2}$ lb. is needed, and 1 lb. when 1 lb. is needed, and so on. The simple fact that it is no easier to drag a body resting on a horizontal floor one way rather than another, shows that the force of friction changes its direction so as to oppose the pulling force. From such experiments we infer the following important rule—The force of friction always acts in such a direction as to prevent motion, if the system is at rest, or to destroy relative motion if the system is moving.

The student will realize that friction is often the cause of motion. Thus, if we place a penny on a card and move the card slowly, the friction will prevent the penny from moving relatively to the card and so will set the penny in motion.

Continuing the experiment with the slab, keep on adding weights to the scale-pan. When the load reaches a certain amount the slab will begin to slide along the table. This shows that there is a certain maximum which the force of friction can attain, and that if a greater force than this is needed to prevent motion, the friction cannot furnish it, and motion will take place.

The fact that there is a limit to the amount of friction which can come into play in any given case is turned to account in many mechanical contrivances known as friction clutches.

A Micrometer Gauge.—B, a fixed nut; C, a screw fitting the nut; A, a milled head turned by the fingers. This head fits tightly on the end of the screw, but is not fixed to it. The reason of this arrangement is that



it is very important that the delicate thread of the screw should not be injured by excessive force being applied to it. If the milled head is turned the friction prevents its slipping on the screw and so the screw turns also, but if the point of the screw comes against a resistance, the friction soon reaches its maximum, and then, instead of forcing the screw into the obstacle or burring the threads of the screw, any further force applied to the milled head

simply causes it to slip round and round without moving the screw.

A similar arrangement is used to prevent the shock of firing a heavy gun from injuring the teeth of the elevating gear.

Our next step is to investigate upon what circumstances the amount of this maximum force of friction depends.

Experiment 2.

Take a thin slab of hard wood—which had better be freshly planed and clean—attach a string to it, and hook a spring balance to the end of the cord. Place the slab on a level table. It will be as well to have two people to carry out the experiment. Place a weight of 2 lbs. on the slab and pull on the spring balance, being careful to keep the string horizontal.

It is good to let the slab have room to move some distance, at least one foot, as at starting a little adhesion, which has been jocularly termed a force of "sticktion," is apt to interfere with the regularity of the results. For this reason it is convenient to just start the slab by hand, and endeavour, by means of the spring balance, to just keep it moving slowly and steadily. Read the spring balance, being careful to avoid parallax.[1] Re-

[1] Parallax is the name given to an error, which may be indicated by the following example :—A pointer moves in front of a vertical scale. The observer in the left-hand figure sees the pointer opposite the graduation 4. The observer in the other figure sees the pointer opposite the graduation 5. But the pointer has in reality not moved at all. It is the observer who has moved. All observations in which the position of a pointer is read from a scale are liable to parallax error. It may be diminished by bringing the pointer very

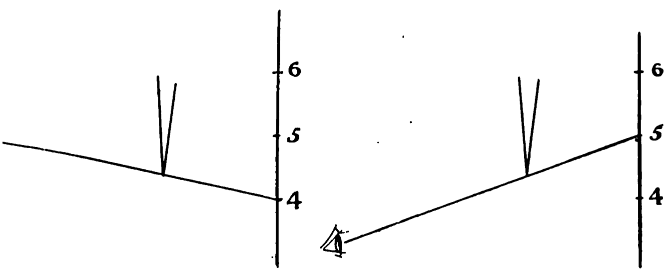
ıt the experiment, with loads on the slab of 4, 6, 8, 10, l 12 lbs. Plot the results of the experiment as a graph.

EXAMPLES.

. In an actual experiment the following values were obtained ight of slab negligible)—

ad of slab in pounds.	Pull on spring balance which just maintains motion.	Value of $\frac{F'}{N}$.
N	F′	
2	0·55	
4	1·10	
6	1·75	
7	2·20	
9	2·65	
11	3·20	
13	3·75	

r to the scale, and by trying to keep the line of sight at right les to the scale. Familiar examples of the effect of parallax may given, *e. g.* (1) the apparent relative movement of objects on ing one eye ; (2) the difficulty of telling the time to a minute



n one looks at a clock from a position very much to one side ; the impossibility of judging a race unless the judge is standing he winning post ; (4) the use of a mirror below the needle of a 'anometer. When the reading on the scale of the galvanometer of reflection is the same as that of the needle there is no parallax error.

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Plot these results as a graph, and you will find that the representative pionts are very approximately in a straight line. What do you infer as to the relation between F′ and N? What value for the ratio $\frac{F'}{N}$ is suggested by the slope of this line?

2. Treat the results of your own experiment in a similar way. You will probably conclude that the maximum possible force of friction F′ between two given surfaces is directly proportional to the force N pressing them together, and that in Example 1, the ratio $\frac{F'}{N}$ has approximately the constant value 0·28.

3. Suppose with a total load of 10 lbs. motion ensues when the pulling force is 2·7 lbs., but that the string makes an angle of 5° with the horizontal. What is the real horizontal force, and the real normal pressure? Compare the true value of $\frac{F'}{N}$ with the value which will be found if the obliquity of the string is not noticed.

We may remark that it is not easy to obtain very consistent results from experiments on friction, even when every precaution is observed.

Always use the same portion of the table throughout the experiment, and be careful not in any way to alter the nature of the surfaces, for instance by polishing them afresh, in the course of the experiment.

In experimenting with wooden surfaces fewer irregularities will be encountered if the direction of motion is along the grain of both surfaces.

The effect of "sticktion" may generally be greatly diminished by keeping up a rapid gentle tapping on the table.

The conclusion that the maximum possible force of friction F′ between two given surfaces is proportional

to the force N pressing them together is expressible by the equation

$$F' = \mu N,$$

where μ is a coefficient.

The numerical value of μ in any particular case depends upon a great many circumstances.

If several members of a class have experimented with different kinds of wood a comparison of their results will show that, although the value of $\frac{F'}{N}$ may be very uniform in each separate experiment, the values of the ratio for different kinds of wood are by no means the same.

The value of μ for two given surfaces is often termed the "coefficient of friction." The value of the coefficient of friction in any given case depends not only upon the nature of the materials in contact, but upon their surface structure (polished, planed, with the grain or across it and so on); and thus an experiment in which the coefficient of friction of, say, oak on oak was determined would only furnish an approximate indication of what to expect even from other pieces of oak.

Does Area of Surface of Contact Affect Friction?—In trying to investigate the circumstances determining the maximum force to friction in any given case it should occur to the student that the *area* of the surfaces in contact may affect the result.

To investigate this by experiment it is best to use polished metal surfaces, but satisfactory results can be

got by using a glass block, sliding on a level surface of paper.

The reason of this is that we wish to vary only one circumstance of our experiment at a time. Any change observed in the maximum friction must then be due to the only condition that has changed.

Now it is impossible to be sure that two pieces of wood differ *only* in area of surface. Their flatness, smoothness, state of grain may differ, but a well-polished block of metal or glass is remarkably uniform in surface.

EXPERIMENT 3.

Place a large sheet of paper or blotting paper on a level table. Take a rectangular slab of glass (in an actual experiment a block $4\frac{1}{2}$ inches by $2\frac{1}{2}$ inches by $\frac{3}{4}$ inch was used), slip a loop of string round it, hook it on to the spring balance and place it on the table. Placing a bit of card or paper on the top of the slab to prevent scratching the glass, put a weight of say 4 lbs. on the block and determine what pull in the spring balance will just move the block.

Repeat this with the block standing on two other unequal faces and also with other weights. You will probably conclude that so far as this experiment shows—the maximum force of friction is independent of the area of the surfaces in contact.

We have already remarked that practically it is not very easy to obtain two surfaces which differ *only* in area; *and it is* also clear that if the pressure is so great as to

alter the character of the surfaces, which is more likely to occur when the area of contact is small, our conclusion will not hold good.

It is important to notice that in all the foregoing experiments the surfaces in contact have been supposed to be dry. No lubricant such as oil or grease has been used. The laws of friction when lubricants are employed are very different from those of solid friction; and, in particular, the maximum friction is not independent of the rubbing area.

Examples.

4. A uniform ladder inclined at 65° rests between a rough pavement and the smooth wall of a house. Find the amount of friction brought into play if the ladder weighs 85 lbs.

5. A 14-lb. weight rests on a plane inclined at 29° to the horizontal. What frictional force is called into play?

Summary.—It may be convenient to bring together the inferences we have drawn.

(1) Friction always opposes relative motion.

(2) The force of friction F will in any given case assume any value, lying between 0 and a certain maximum F′ which may be required to prevent motion.

(3) The maximum force of friction F′ is equal to μN where μ is a coefficient whose value depends on the material and the condition of the surfaces in contact, and N is the pressure between the surfaces, at right angles to them.

N is generally termed the normal pressure.

EXAMPLE.

6. If there were no friction what would happen to books or dishes placed on a table when the table was moved horizontally?

Smooth Surfaces.—Two surfaces between which no friction can act are termed smooth. No *perfectly* smooth surfaces exist, but fine ice or highly polished metallic surfaces approximate towards perfect smoothness.

If we do not wish to rely upon friction as a means of preserving equilibrium we assume that the bodies are smooth; and if the other forces acting can then preserve equilibrium they will certainly do so when the bodies are rough.

If we desire to *prevent* slipping between two surfaces we may either—

(*a*) Press them more strongly together.

(*b*) Increase the coefficient of friction between them.

Examples: (*a*) A tight belt is less likely to slip on a pulley than a loose one.

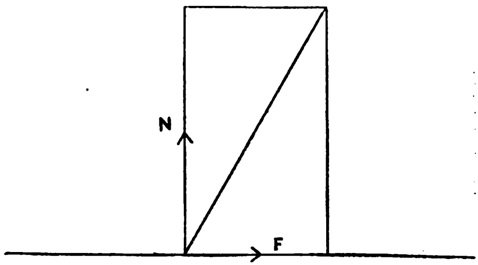
In a locomotive engine, the designer tries to concentrate weight on the driving wheels.

It is said that if a horse drawing a cart on a frosty day cannot move because his hoof slips, increasing the load *on his back* will sometimes enable him to do so.

(*b*) Scattering sand on a frozen road, or on the rails when they are wet or greasy.

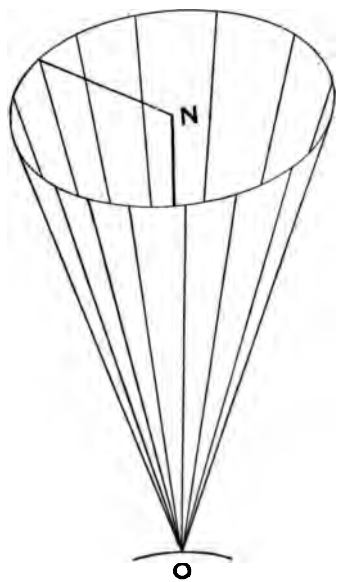
Total Reaction.—The resultant of the friction **F** and the normal pressure N is often termed the *total reaction* R of the surface.

The total reaction R makes with the normal pressure



N an angle whose tangent is $\frac{\text{F}}{\text{N}}$. As the maximum value of F is F′, and as $\frac{\text{F}'}{\text{N}}$ is the coefficient of friction we see that the greatest possible angle a between R and N has its tangent equal to μ, that is—

$$\tan a = \mu.$$



This angle is often termed the "angle of friction."

The angle of friction is the angle whose tangent is equal to the coefficient of friction.

The line of action of R may lie anywhere within a right circular cone—vertex at the point of contact, axis along the normal, and semi-vertical angle a, but cannot lie outside this cone.

EXAMPLES.

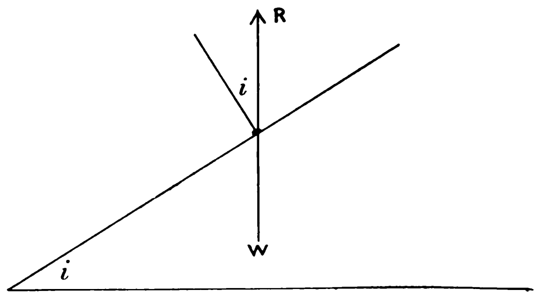
7. Express the maximum total reaction R′ in terms of N and α.

8. A uniform cubical block rests on a horizontal table, and is just not moved by a horizontal force applied to the top edge, parallel to the sides. At what point of the base does the resultant total reaction act? Find whether the block will commence to upset or slip if $\alpha = 20°$, and if $\alpha = 30°$.

9. A small flat disc is placed on the outer surface of a rough sphere. If the angle of friction is α, draw a diagram showing the portion of the surface upon which the disc can rest without slipping.

10. A number of equally rough blocks are piled symmetrically one on another, and a gradually increasing horizontal force is applied to the centre of a vertical face of the topmost block. If equilibrium is broken by sliding taking place, how will movement occur? If equilibrium is broken by the system upsetting, how will movement occur?

The fact that, when the friction is at its maximum the direction of the total reaction makes an angle, whose tangent is μ, with the normal to the surface, often enables us to simplify the solution of a problem.



Thus, consider a stone resting on an inclined plane. The forces on it are: (1) The weight W vertical; (2) the total reaction, R, of the plane. As the stone is at rest, these forces must be equal and opposite; that is, the

total reaction must be vertical. Now the angle between the vertical and the normal to the plane is equal to the angle inclination i of the plane.

Hence we infer: If a stone is at rest on an inclined plane the inclination of the plane cannot exceed the angle of friction.

EXAMPLES.

11. If a stone will just not remain on a plane whose inclination is 14°, what is the angle of friction, and what is the coefficient of friction?

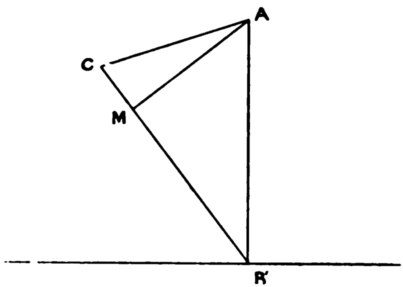
12. A block of stone lies on a plank. The plank is gradually tilted, and the stone slips when the plank is inclined at 25° to the horizontal. What is the co-efficient of friction between the block and the plank?

13. A straight uniform plank, 20 feet long, and weighing 100 lbs., rests with its middle point upon a rough horizontal cylinder of 2 feet radius, and at right angles to the axis of the cylinder. If the angle of friction is 45°, find the greatest weight that can be suspended from one end of the plank without upsetting it.

14. Three uniform rods, AB, BC, CD, each 2 feet long, are rigidly fastened together so as to form three sides of a square. The frame thus formed is hung up over a rough peg E (coefficient of friction $\frac{1}{4}$), so that BC is the rod in contact with the peg. Show that the distance of E from the centre of the rod BC cannot exceed 2 inches if equilibrium is to be preserved.

As another example, let us now consider a block resting on a horizontal plane, which is to be dragged along by a rope. The forces on the block when it is just on the point of moving are: (1) The weight of the block W (vertical); (2) the pulling force P, which we will suppose makes an angle θ with the horizontal plane; (3) the total reaction R. The block is on the point of moving, so the friction is at its maximum, and hence the direc-

tion of R must make an angle a, equal to the angle of friction, with the vertical.



Draw a vertical line, AB, to represent W. Through A draw a line parallel to the direction of P, and through B a line parallel to that of R, these lines meeting in C. Thus the angle A of the triangle ABC is $90^\circ - \theta$, and the angle B is a. As the sum of the three angles of the triangle is 180°, C must be equal to $90^\circ - a + \theta$.

Now
$$\frac{P}{W} = \frac{AC}{AB}$$

Hence, if the direction of the pull P alters, the magnitude of P changes in the same way that AC changes.

Hence, if we draw a perpendicular AM on to BC, we perceive that AM is the direction of the *least* pull that will move the block and its magnitude P′ is given by the equation—

$$\frac{P'}{W} = \frac{AM}{AB} = \sin a.$$

Notice that we have—

$$\frac{P}{W} = \frac{AC}{AB} = \frac{\sin a}{\sin (90 - a + \theta)}$$
$$\frac{R}{W} = \frac{BC}{AB} = \frac{\sin (90 - \theta)}{\sin (90 - a + \theta)}$$

Examples.

15. Find the least force which will drag a 250-lbs. sledge along when $a = 5^\circ, 10^\circ, 15^\circ, 20^\circ, 25^\circ, 30^\circ, 35^\circ, 40^\circ, 45^\circ$, respectively.

16. A 250-lbs. sledge is dragged along by a force applied at an inclination θ to the horizontal. Make a table of corresponding values of P and θ, taking (1) $a = 10^\circ$ and $\theta = 0^\circ, 5^\circ, 10^\circ, 15^\circ, 20^\circ$; (2) $a = 30^\circ$ and $\theta = 0^\circ, 10^\circ, 20^\circ, 30^\circ, 40^\circ$.

17. A block of stone weighing 500 lbs. rests on a plane, whose angle of inclination is i. The angle of friction is a. A force P is applied, at an angle θ with the horizontal which just drags the stone straight up the plane, that is along "a line of greatest slope." Draw a triangle of forces and hence find the value of θ corresponding to the least possible value of P.

18. In the preceding example suppose the block is to be dragged *down* the plane. Find the direction and magnitude of the least possible force.

[Notice that, as friction always opposes relative motion, the direction of the friction is reversed when we try to drag the block down and that consequently the total reaction R now makes an angle a with the normal to the plane on the other side.]

19. A block of stone weighing 400 lbs. lies on a floor, and is to be moved by a rope attached to it. The angle of friction between the stone and floor is 35°. Determine graphically the force required to move the stone when the rope is pulled (1) horizontally, (2) at an inclination of 20° to the horizontal, (3) at an inclination of 40° to the horizontal.

By general reasoning, when the angle of friction is a°, determine the direction in which the rope must be pulled to make the force required to move the stone as small as possible.

20. A weightless string, with a weight P hanging at one end, passes over a small pulley and is attached to a small ring, also of weight P sliding on a rough vertical wire. Show that the ring can rest at any point of the wire where the upper angle between the string and the wire is less than twice the angle of friction.

21. A packing case of weight 7 cwts. is hauled up a plane inclined at 40°. Find the work done if the coefficient of friction is 0·6 and the plane is 25 feet long. If released at the top is the friction sufficient to prevent it sliding down?

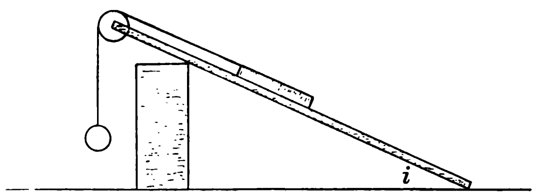
22. A box weighing 450 lbs. is slowly hauled up a rough inclined

plane of length 30 feet, the top of the plane being 12 feet above the bottom. If the amount of work expended is 3 foot-tons, find the coefficient of friction for the box and the plane.

23. A box, weighing 240 lbs., is supported on a plane inclined at 40° to the horizon by a rope attached to it and inclined at 20° to the plane. If the coefficient of friction is 0·4, what is the least pull in the rope required to move the box up the plane?

Experiment 4.

Coefficient of Friction by Inclined Plane.—A plane is inclined at a certain angle i to the horizon and a block of wood slides on the plane being attached by a thread which passes over a smooth moving pulley fixed at the top of the



plane, to a scale-pan. The thread should be attached to a point on the block so that it is *parallel to the plane.*

The methods of experimenting are as follows: (1) The plane is fixed at a definite inclination, which should be considerably greater than the angle of friction, and the least force which will pull the block up the plane is found by putting weights into the scale-pan. Weights of various magnitudes are put on the block, and in each case, the least force requisite to pull the block up the plane is determined. (2) In a similar manner we find the least force which will keep the block from sliding down the plane.

An approximate value of the coefficient of friction can be obtained by finding the inclination of the plane at which the block will just slide down the plane of its own accord.

In the following experiment the block and the plane were both of wood and the direction of motion was with the grain of both block and plane.

The block slid of its own accord when the angle of inclination of the plane was about 15°. This angle was, in the experiment, measured by an inclinometer, but there is no difficulty in getting the angle of inclination with very fair accuracy with an ordinary six-inch semi-circular protractor. (Notice that if a plummet is suspended from any point of the plane the angle it makes with the plane is the complement of the required angle.)

(1) Weight of block itself 1·1 lbs. Weight of scale-pan with suspending chains = 0·19 lb. $i = 27° 8'$.

P Least force required to move the block (including weight of scale-pan).	W Weight of block and additional load.	$\frac{P}{W}$
1·5 lbs.	2·1 lbs.	0·714 [1]
2·1 „	3·1 „	0·678
2·9 „	4·1 „	0·707
4·2 „	6·1 „	0·689
		Sum 2·788

[1] We keep *three* significant figures in the value of $\frac{P}{W}$ as we are going to take the arithmetic mean of four different values.

$$\text{Average value of } \frac{P}{W} = \frac{2{\cdot}788}{4} = 0{\cdot}697.$$

In this case the equation giving the relation between P and W is—

$$P = W \sin i + \mu W \cos i,$$

or
$$\mu = \frac{P}{W} \sec i - \tan i.$$

Here $\tan i = 0{\cdot}527$, $\sec i = 1{\cdot}13$—

$$\therefore \mu = 0{\cdot}697 \times 1{\cdot}130 - 0{\cdot}527,$$
$$= 0{\cdot}788 - 0{\cdot}527,$$
$$= 0{\cdot}261.$$

(2)—

P Least force required to keep the block from moving (including scale-pan).	W Weight of block and additional load.	$\frac{P}{W}$
0·45 lbs.	2·1 lbs.	0·214
0·73 „	3·1 „	0·235
0·96 „	4·1 „	0·234
1·49 „	6·1 „	0·244
		Sum 0·927

$$\text{Average value of } \frac{P}{W} = \frac{0{\cdot}927}{4} = 0{\cdot}232.$$

The equation in this case is—

$$P = W \sin i - \mu W \cos i,$$
$$\mu = \tan i - \frac{P}{W} \sec i.$$

In this experiment also $i = 27^\circ\ 8'$

$$\therefore \mu = 0{\cdot}527 - 0{\cdot}232 \times 1{\cdot}13$$
$$= 0{\cdot}265.$$

The inclination of the plane at which the block slid down of its own accord was 15°, and since $\tan 15^\circ = 0{\cdot}268$ we have another experimental value for μ. This last value, however, could not be depended on as it was very difficult to ascertain the exact angle at which slipping took place.

EXAMPLES.

24. Would it be easier to pull a packing-case weighing 200 lbs. up a plane inclined at 28° to the horizon (coefficient of friction 0·32) or to lift it directly by means of a rope?

25. If the inclined plane in Example 24 is 20 feet long compare the work done in the two cases.

When a body is at rest on a surface the total reaction of the surface is equal in magnitude and opposite in direction to the resultant of all the other forces on the body.

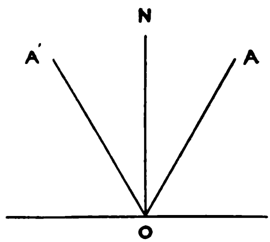
Now, until the breaking point is reached, there is no limit to the amount of normal pressure which a surface can exert, and consequently there is no limit to the *magnitude* of the total reaction.

But, as we have already seen, there is a limit to the *direction* of the total reaction.

Suppose ON the normal to the surfaces, and draw OA, OA′ (each making an angle equal to the angle of friction, with ON).

(i) Suppose the resultant of the other forces falls inside the angle AOA′. The total reaction can act in the same

line, and as its magnitude is unlimited it can certainly balance the other forces (provided of course their resultant presses the surfaces together). Thus if the other forces have a resultant acting towards O within the angle AOA′ the body will never slip on the surface, however great the forces applied may be.

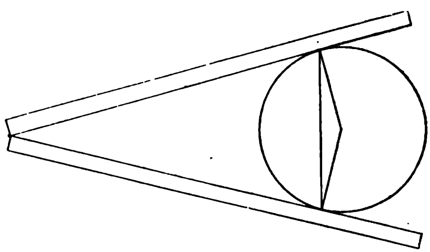


(ii) If, on the other hand, the resultant of the external forces acts outside the angle AOA′ the total reaction cannot possibly balance the other forces, and slipping will at once take place.

EXAMPLES.

26. A slab of wood placed on a table is pushed by the force P applied at an angle θ to the vertical. Show that if $\mu = 0{\cdot}5$ the slab will not slip, however great P may be, unless tan θ is greater than $\frac{1}{2}$.

27. Two vertical boards are hinged together, forming a **V**. Show that it will be impossible to squeeze out a glass marble, diameter 1

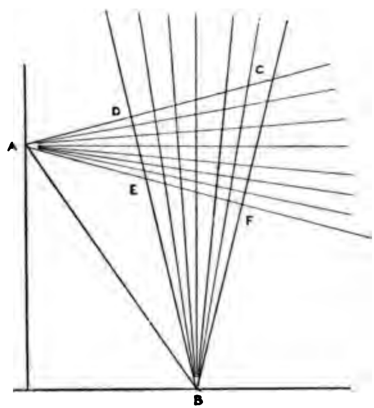


inch, placed between them; unless its distance from the hinge is greater than $\dfrac{1}{2\sin\alpha}$, where α is the angle of friction.

[Hint: Let A and B be the points of contact. Shade the portion of the figure within which the total reaction at A must lie and within which the total reaction at B must lie. Can each reaction act on the line AB? If so they can balance one another however large they are.]

28. A rod AB, whose weight may be neglected, rests against a rough horizontal floor and a rough vertical wall, the angle of friction in each case being 15°, and the inclination of the rod 55° to the horizontal. Between what points of the rod may (*a*) a vertical force, (*b*) a force inclined at 45° to the vertical, be applied without producing motion?

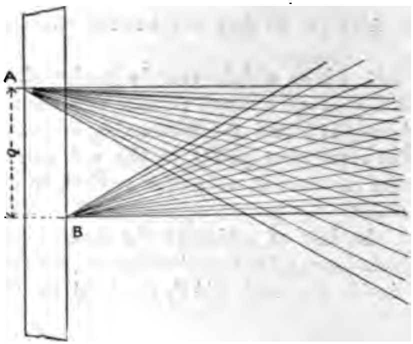
[Notice that the line of action of the applied force must pass through the shaded area CDEF in the diagram, for the total reaction at A must lie inside the angle CAF, the total reaction at B inside



the angle CBD, and the line of action of the third force must, if equilibrium is possible, be concurrent with the lines of action of these total reactions.]

29. A ladder weighing 80 lbs. rests against a rough floor and a rough wall (angles of friction 20° and 15° respectively), and is inclined at 45° to the horizontal. A man weighing 200 lbs. commences to ascend the ladder. Examine whether it will slip before he reaches the top.

30. A bracket can slide on a vertical rod, diameter a, being in contact with it at points A and B (see figure). Show that if it



does not slip the centre of gravity of the load must be at a distance greater than $\dfrac{b - a\mu}{1 + \mu}$ horizontally from B.

31. A uniform rod AB a foot long rests inclined at an angle of 30° to the horizon with the end A on a rough horizontal plane, and is fastened by a string BC to a point C two feet vertically above A. If the rod is on the point of slipping at A, find the coefficient of friction at A.

32. A uniform rod 25 inches long is tied by a string 1 foot long to a point fixed 19 inches above a table, on which the corner end of the rod rests. The angle of friction is 30°. Draw a diagram showing the rod in various positions. Measure in each case the angle which the total reaction makes with the vertical, and hence find by trial the extreme positions of equilibrium of the rod. Construct a table showing the horizontal distance of the foot of the rod from the fixed point and the corresponding inclination of the total reaction.

Distance.	Angle.
In inches.	
18	
19	
20	
21	
22	

33. A rectangular window frame ABCD, weight W, slides in vertical guides being counterpoised by weights, each $\frac{W}{2}$, attached to cords passing over smooth pulleys and attached to the upper corners AD of the frame. If one cord breaks find the condition that the frame may remain supported.

[Hint: If the cord attached to A breaks the frame will turn a very little, and may be considered to be in contact with the guides at A and C. Draw diagrams showing the possible directions for the total reactions at A and C.]

Consider the cases in which AB = 2 feet, BC = 3 feet 6 inches, $\mu = 0{\cdot}5$ or $0{\cdot}6$.

34. A drawer 2 feet wide by 1 foot deep has two handles 15 inches apart. Will it be possible to pull it out by one handle if $\mu = 0{\cdot}4$, and if $\mu = 0{\cdot}7$?

35. A drawer 2 feet wide is opened by two handles placed 16 inches apart. Find the least depth of the drawer if it is possible to pull it out by one handle, if $\mu = 0{\cdot}5$, and if $\mu = 0{\cdot}7$.

36. A weight of 100 lbs. rests on a rough horizontal plane, the angle of friction is 32°, what horizontal force will just start the body? If any smaller force can move it, show how to find it, and determine the direction of the least force which will do so.

37. A weight of 1 ton is dragged 100 feet up a plane whose inclination is 15°, the angle of friction being 12°, and the force applied parallel to the line of greatest slope. Calculate the total work done, the work expended in overcoming the friction, and the work done in raising the weight. What is the "efficiency" of the method [1]?

38. If, in the previous example, the force were applied in the most advantageous manner, what would be the efficiency?

[It is thought that the builders of Stonehenge and other primitive temples made use of inclined planes of earth, which were subsequently removed, for the purpose of raising the huge stones into their place.]

39. To raise a barrel weighing 200 lbs. from a cellar, an inclined plane is used, sloping at 40° to the horizontal. Two ropes are made fast to the upper end of the plane, carried down the plane, round the barrel, and back to the top of the plane. Two men haul on the ropes parallel to the plane, each exerting a pull of 40 lbs. What

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[1] The object being simply to raise the weight through the given vertical height.

work do they do in raising the barrel 9 feet vertically, and of this work how much is stored up in the raised barrel and how much wasted? What is the efficiency of the system?

40. An isosceles triangular lamina stands with its base on a rough horizontal plane, and a gradually increasing horizontal force is applied at the vertex. Show that the lamina will slip before it turns if the angle of friction is less than the semi-vertical angle of the triangle.

41. A beam is lying directly across an inclined plane, the face in contact with the plane being 4 inches across, and the thickness of the beam 3 inches. Show that, if the angle of friction between the plane and beam is 53°, when the plane is gradually tilted the beam will slide before it rolls.

42. A uniform rectangular block rests on a rough horizontal plane. An equal, equally rough, block is placed exactly on the first, and a gradually increasing horizontal force is applied to the centre of a face of the upper block, and at right angles to it. Find whether the upper block will slip on the lower, or whether both blocks will begin to upset, if $\mu = \cdot 5$.

43. If the angle of friction in Ex. 37 is such that the block if released will remain at rest on the plane and not slide down, the efficiency cannot be so great as one half.

Efficiency of Machine with Friction.—It has been already stated that it is often desirable that a weight if released should simply remain at rest, and not "take charge."

Example 43 illustrates a general principle, namely, if in any machine for raising weights the frictional forces are independent of the lifting force—then if the machine is such that the weight will not descend if the lifting force ceases to act, its efficiency cannot exceed 50 per cent.

For, let a force P, doing work of amount Pl, raise the weight W through a vertical height h doing work of amount K foot-pounds against frictional resistances.

Then $$Wh + K = Pl.$$

The weight W, descending through h can do Wh foot-pounds of work. As the friction is independent of the lifting force, K foot-pounds of work must be expended, in the descent, in overcoming friction. Therefore, as the weight will not descend by itself, K must be greater than Wh.

In fact, we shall have—

$$\mathrm{K} - \mathrm{W}h = x,$$

where x is the number of foot-pounds of work which must be done by outside help to draw the weight down a distance h. Our equations are—

$$\mathrm{K} + \mathrm{W}h = \mathrm{P}l,$$
$$\mathrm{K} - \mathrm{W}h = x.$$

$$\text{Efficiency for raising W} = \frac{\mathrm{W}h}{\mathrm{P}l} = \frac{\mathrm{W}h}{2\,\mathrm{W}h + x} = \frac{1}{2 + \dfrac{x}{2\,\mathrm{W}h}},$$

which is always less than one half.

In practice, a machine is often so designed that additional frictional forces come into play when the lifting force ceases to act. Thus, in many lifts if the rope broke cams would wedge the sides of the lift against the walls of the shaft and so check its descent.

Newton's First Law.—We have already seen that there can be no *resultant* force acting on a body which is at rest. If we apply an additional force to a body which is moving we make it move faster, or slower, as the case may be. If, then, a body is moving uniformly, neither faster nor slower, it is acted on by no *resultant* force.

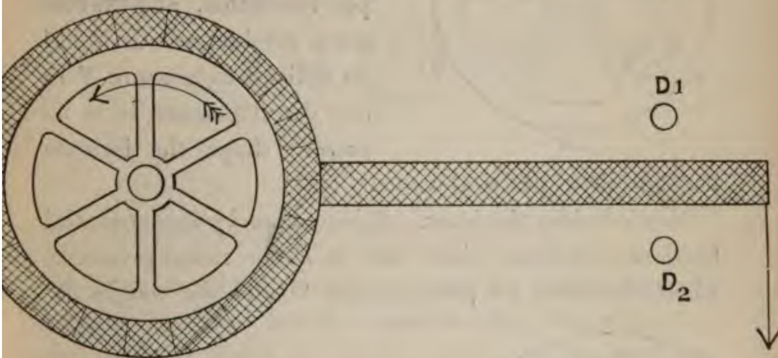
For instance, the pull of the engine on a train tends to make it go faster. The resistance of the air and the friction tend to make it go slower. If the train is gaining speed the pull of the engine is greater than the total resistance, and there is a resultant forward force. If the train is losing speed (as when encountering a very strong head-wind) the resistances are greater than the pull of the engine, and there is a resultant backward force, which, if the pull of the engine remained constant, would ultimately bring the train to rest. If the train is neither gaining nor losing speed, but running at a perfectly steady rate, the pull of the engine is exactly equal to the total resistance. This conclusion was expressed by Newton in his first law of motion—

"Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum nisi quatenus illud a viribus impressis cogitur statum suum mutare."—"Every body continues in its state of rest or of uniform motion in a straight line save so far as it is compelled by external forces to change that state."

Friction Couples.—Again, imagine a vertical wheel which can turn about a horizontal axis through its centre of gravity. A couple continually acting on the wheel in its plane will make it turn faster and faster. A friction couple, on the other hand, continuing to act on the revolving wheel will check its speed. If, then, the wheel is spinning uniformly, neither gaining nor losing speed, the "accelerating" couple must exactly balance the "retarding" couple.

Dynamometers.—This principle is turned to account in the so-called "dynamometers" for measuring the work done by an engine.

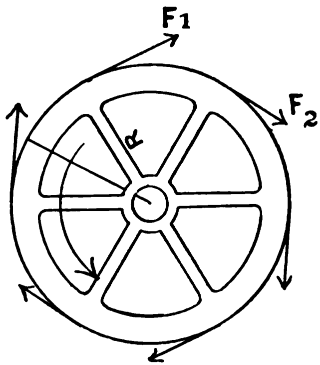
In the Prony brake[1] the engine turns a fly-wheel or pulley revolving on a fixed centre. A collar grips the circumference of this wheel somewhat tightly, and a lever projecting from the collar, as in the figure, can have



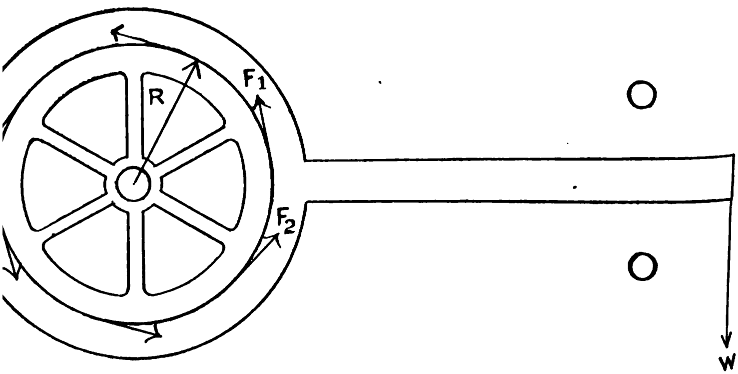
weights hung from it at the end. Two "stops," D_1 D_2, are merely to prevent the lever from flying round. When the wheel turns the friction between it and the collar will raise the lever and bring it against D_1. Weights are hung on until the wheel runs at a steady speed—with the lever between the stops, not touching either. Consider the forces acting on the wheel, viz. the couple due to the engine, the frictional resistances, F_1 F_2 . . ., tangential to the rim, and the radial pressures between the rim and collar. The forces of friction all round the rim give a couple whose moment

[1] So called from its inventor, Gaspard de Prony (1755–1839), a celebrated French engineer.

about the centre O is Fr, say, F being the sum of all the tangential frictional forces. Therefore, as the wheel is running steadily, the engine is supplying an equal "accelerating" couple, and so doing work of amount $2\pi Fr$ per revolution, and $2\pi Frn$ per n revolutions. It would be difficult to measure F in any direct manner, so in its present shape the formula has but little practical value.



Now consider the lever. It is acted on by the tangential frictional reactions whose sum is F, the radial pressures, whose directions all pass through O, and the weight W.



Its fulcrum is O, and taking moments about O we have, if a is the distance of the line of action of the weight from O $Wa = Fr$. Therefore the work done by the engine is

$2\pi n\mathrm{W}a$ in n revolutions, so if the wheel is turning n times per minute, the work done per minute is $2\pi n\mathrm{W}a$, and the horse-power[1] $=\dfrac{2\pi n\mathrm{W}a}{33000}$.

In an actual experiment with a motor, W was 7·6 lbs., the motor ran at a steady speed making 143 revolutions in half-a-minute, the arm of the lever being 25 inches. The result was

$$\text{H.P.} = \frac{2\pi \times 7{\cdot}6 \times \frac{25}{12} \times 286}{33000},$$
$$= 0{\cdot}86.$$

Example.

44. How should the weight of the lever be allowed for?

It is an excellent exercise to draw the foregoing piece of mechanism in different colours, showing, for example, fixed points in black, the wheel in red, and the lever in blue.

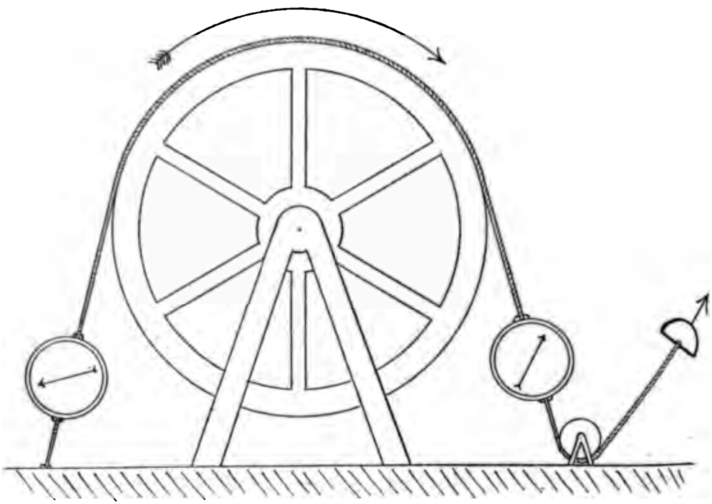
A variation of the preceding method is shown in the diagram below.

A rope is attached to a fixed eye and has a spring balance attached to it. To the hook of the spring balance is fastened a rope which passes over the wheel, and is fastened to the hook of another spring balance. A rope fastened to the latter passes through a ring, and can be

[1] The rate at which an engine can do work is measured in horse-powers. An engine which does 33,000 foot-pounds of work in one minute is said to be working at 1 horse-power. A 12 H.P. motor car has an output of 12 × 33,000 = 396,000 foot-pounds of work in one minute.

pulled taut by a handle. When the engine runs steadily the readings, T_1 T_2, of the spring balances are taken.

Consider the forces[1] acting on the part of the rope in contact with the wheel and take moments round O. Consider the work done by the engine per revolution.



Hence show that the horse power of the engine when running at n revolutions per minute is $\dfrac{2\pi n(T_1 - T_2)r}{33000}$.

It will be noticed that in the foregoing methods the

[1] The forces acting are (1) the pulls T_1, T_2 on the ends; (2) certain radial pressures—the way these vary from point to point along the rim is unknown, but they all pass through O; (3) certain tangential frictional forces—the way these vary from point to point is unknown, and for our purpose immaterial, for if their arithmetical sum is F, their moment is Fr, r being the radius.

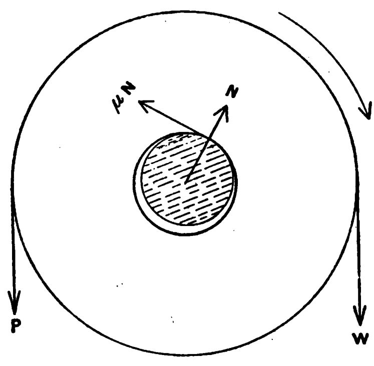
engine can do no useful work while the test is being applied.

The Friction of an Axle in a Circular Bearing.—(1) Suppose the axle fits loosely in the bearing. Axle and bearing are then in contact over a small area which may be treated as a point.

EXAMPLE.

45. A pulley of radius b can turn on an axle of radius r. Find the least vertical pull on a rope passing over the pulley which will just support a weight W.

The forces acting on the pulley are the pulls P and W, and the radial pressure N of the axle and the force of friction, which as motion is about to ensue is μN.



Suppose the radius to the point of contact makes an angle a with the vertical. The radius of the aperture is supposed only just to exceed r. The total reaction R, resultant of N and μN, must be vertical. (Why?) Consequently we must have $\mu = \tan a$, where a is the angle of friction. We now have—

revolving vertically $\qquad P + W = R.$

taking moments round the centre—

$$Pa + Rr \sin a = Wa.$$

Substituting $\quad Pa + (P + W) \sin \alpha \,.\, r = Wa,$

or
$$P = W\frac{a - r\sin\alpha}{a + r\sin\alpha}.$$

Examples.

46. If $W = 100$ lbs., $a = 4$ inches, $r = \frac{1}{4}$ inch, $\alpha = 20°$, what is the least vertical force which would just support W?

47. With the same data what is the least vertical force which will just raise W?

48. What is the least horizontal pull P which will (*a*) just support the weight, (*b*) just raise the weight?

When an axle revolves in a bearing it is usually lubricated. The laws of fluid friction are quite different from those of solid friction: and consequently examples such as the foregoing are of but limited application.

(2) The axle fits tightly in the bearing, so that there are radial pressures, and tangential frictional forces all round the axle. A compass joint or a jointed carpenter's rule furnish instances.

49. A uniform rod, weight W, length $2a$, can turn about a horizontal axle radius r, whose centre is at a distance b from one end. If it can remain in a horizontal position, what couple must the friction of the axle furnish?

50. A 2-feet carpenter's rule (which may be considered as uniform) is partly opened and stands upright on a table. Find the frictional couple acting at the joint if the rule weighs 4 ounces, is an inch broad and is opened at an angle of 100°.

51. Two equal uniform rods, length $2a$, weight W, hinged together, are placed astride of a smooth horizontal cylinder radius r. Show that the couple at the hinge necessary to maintain equilibrium when each rod makes an angle α with the vertical is

$$W\,(r \cot\alpha \operatorname{cosec}\alpha - a \sin\alpha).$$

52. In a wheel and axle, the axle rests in rough bearings (angle of

friction ϵ). Show that the least force which, acting vertically downwards, will raise a weight W is $\frac{b(1+\sin\epsilon)}{a-b\sin\epsilon}$ W, where a, b are the radii of the wheel and axle respectively.

53. In the preceding example find the least force applied horizontally which will raise W.

54. A barrel is being dragged up an inclined plane (coefficient of friction μ) by a rope coiled round the barrel and leaving it horizontally. Find the inclination of the plane if the barrel is just about to slip.

55. Explain why the brake of a bicycle is usually applied to the rim or tyre and not to the hub.

56. A rotating wheel is stopped in four complete revolutions by a frictional force of 5 lbs., applied at a distance of 2 feet from the axle. If a 10-lb. frictional force were applied at 6 inches from the axle, how many revolutions would the wheel make before coming to rest?

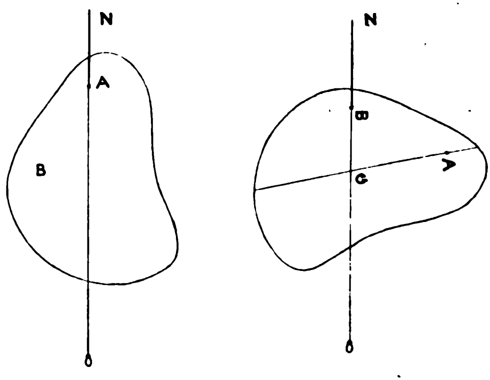
57. A 2-feet carpenter's rule, weighing 5 ounces, hinged at its centre, is opened, and laid in a horizontal position across two pegs, one at an end of the rule. Find the position of the other peg if the joint is perfectly free, and find limits for the position of the second peg if the joint is stiff and capable of exerting a frictional couple of 10 ounce-inches.

58. A circular shaft of a foot radius turns in a horizontal bearing. The shaft is an easy fit in the bearing, so that contact occurs only along a horizontal line. The vertical force thrusting the load down on its bearings is W tons, and the angle of friction is ϕ. Find the position of the horizontal line of contact, and show that the retarding moment due to friction is a W sin ϕ tons-feet.

CHAPTER VII

CENTRE OF GRAVITY

Any body may be supposed to be made up of a large number of particles, the weights of these particles form a number of parallel forces, and the resultant of all these parallel forces may be found by the rules already established. This resultant force which in magnitude is equal to the sum of the weights of the particles will



always pass through some point fixed relatively to the body (see p. 115) in whatever position the body may be.

The centre of gravity of a rigid body is that point fixed relatively to the body through which the line of action

of the resultant of the weights of all the particles composing the body always passes in whatever position the body may be.

Finding the centre of gravity of a body depends on finding the line of action of the resultant of a set of parallel forces; and the student who understands the method of doing this by taking moments should have no difficulty now.

Experiment 1.

The position of the centre of gravity (hereafter abbreviated to C.G.) of a thin flat plate of metal or piece of cardboard can easily be found experimentally.

Make a small hole near the edge of the plate and suspend the plate by a thread from a nail. The plate will rest in such a position that its C.G. is vertically below the nail, and the thread will be vertical, for the only two forces acting on the plate are the tension of the thread and the weight of the body, and two forces to be in equilibrium, must have the same line of action, be equal in magnitude, and opposite in direction. We thus infer that the C.G. lies in the vertical line through the nail—that is to say, in the prolongation of the thread. The position of the line can be marked on the plate. So if we suspend the plate from any other point on its rim, a second line can be drawn in which the C.G. will lie. The point of intersection of the two lines will be the required C.G. As a verification of this position other lines may be drawn, in which the C.G.[1] should lie.

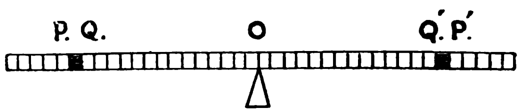
[1] Strictly speaking the C.G. of a uniform card or plate is of course at the middle point of the line joining the centres of gravity of the two faces, found by the foregoing construction.

The C.G. of a few bodies of exact geometrical form can be ascertained by considerations of symmetry and sometimes by geometrical devices.

We first consider flat plane bodies. Such bodies are of course fictions of our imagination, for all material bodies have thickness as well as length and breadth, but the thickness may be very small, as in the case of the piece of card we have already dealt with experimentally. Besides we can frequently find the C.G. of a solid body like a pyramid by splitting it up into thin parallel plates—the Latin word lamina (= plate) is frequently used in this connection. When we speak of the C.G. of a triangle or quadrilateral we mean the C.G. of a thin plate (like a uniform coat of paint) which has the form of the triangle or quadrilateral.

Uniform Rod.—The C.G. of a uniform straight rod of inappreciable thickness lies at its middle point. We have hitherto taken this as self-evident, as we know that such a rod will balance if supported at the centre.

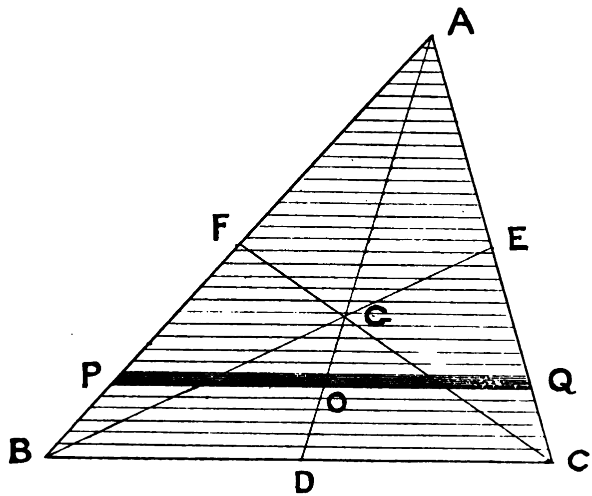
We may, however, realize this more clearly by supposing the rod divided into a large number of equal bits. Any



two of these equal bits at equal distances from the centre, such as PQ and P′Q′ have their C.G. at O, the middle point of the rod, and therefore O is the C.G. of the whole rod.

Centre of Gravity of a Triangular Lamina.—Divide the

triangular plate by lines parallel to the base BC into a number of thin strips, then we know from elementary geometry that the centres of all these thin slips lie in the straight line joining A to D, the middle point of BC. But the C.G. of any one strip, such as PQ, lies at O its centre, and therefore the C.G. of the whole triangle must



lie somewhere in the median AD. Similarly by dividing the triangle into strips parallel to AC, we can show that the C.G. lies in the median BE. The point of intersection G, of AD and BE will therefore be C.G. of the triangle.

Since the C.G. must also lie in third median CF, it follows that the three medians of a triangle meet in one point. It will be noticed that we have arrived, by the aid of mechanical principles, at the well-known geometrical

theorem, "The three medians of a triangle are concurrent." Many other instances of similar "statical" proofs of geometrical theorems are to be found in mechanics.

EXAMPLES.

1. By considering how to construct the C.G. of weights l, m, n placed at the angular points A, B, C of a triangle prove that if concurrent lines are drawn through the angular points A, B, C of a triangle meeting the opposite sides in D, E, F then

$$BD.CE.AF = CD.AE.BF.$$

[The geometrical theorem is known as Ceva's theorem: having been published in 1678 by Giovanni Ceva.]

Calling the C.G. G, it is easy to prove that all the six triangles which have their vertices at G are equal. Thus to prove that the $\triangle$ AGF = $\triangle$ DGC: Since BD = DC, the $\triangle$ ABD is half of the whole $\triangle$ ABC, and since BF = FA, the $\triangle$ BFC is half of the whole $\triangle$ ABC, therefore the $\triangle$ ABD = $\triangle$ CBF, and therefore taking away the common part GFBD, the $\triangle$ AFG = $\triangle$ DGC. Similarly it may be shown that any two of the six triangles are equal.

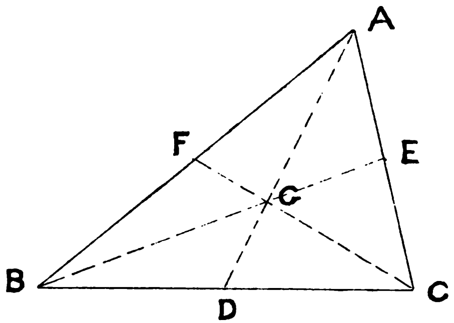
Since the triangle ABG is twice the triangle BGD, therefore AG = 2GD; that is to say, the C.G. of a triangle divides the line joining the vertex to the middle point of the base in the ratio 2 : 1.

2. A square is divided by its diagonal into two parts and one of these is hung up by a string attached to one of the acute angles. Prove that the opposite side makes an angle 26° 33′ with the horizon.

Equivalent System for a Triangle.—The result just stated enables us to prove very easily that the C.G. of the

three particles of equal weight, placed at the angular points of the triangle, coincides with the C.G. of the triangle itself.

The resultant of two weights each equal to W at B and



C is a weight 2W at D the middle point of BC. And weights of 2W at D and W at A have as their resultant a weight of 3W at G where AG : GD :: 2 : 1. In other words, the C.G. of the three equal weights coincides with the C.G. of the triangle.

The property just proved is useful in dealing with problems involving the weight of a triangular lamina.

EXAMPLE.

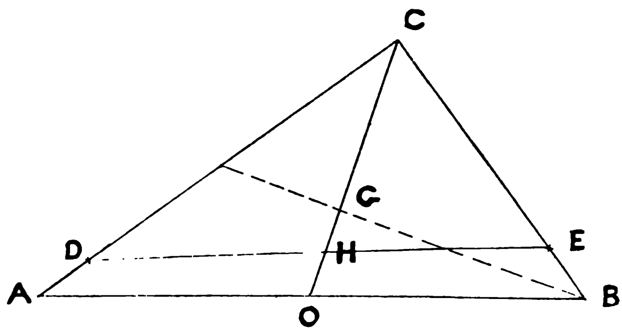
3. A heavy triangle ABC has AC $3\frac{1}{2}$ inches in length and BC $2\frac{1}{2}$, the angle C being a right angle. It is supported on three pegs at C, D and E (see below), where AD = BE = $\frac{1}{2}$ inch. Find the pressures on the three supports.

Let P, Q, R be the pressures on C, D and E. If W denotes the weight of the triangle we may replace W by three

forces each $\frac{W}{3}$ at A, B and C. The three forces P, Q and R balance these three equal weights. Taking moments about BC (see p. 124), we have—

$$\frac{W}{3} AC = Q . DC,$$

$$Q = \frac{W}{3} \times \frac{3{\cdot}5}{3} = \frac{7W}{18}.$$



Examples.

4. Find R by taking moments about AC, and deduce the value of P.

5. Solve Example 3 thus: Draw the triangle to scale, draw the median CO. Find the C.G. G and join DE, meeting CO in H. Prove that—

$$P = W \frac{GH}{CH},$$

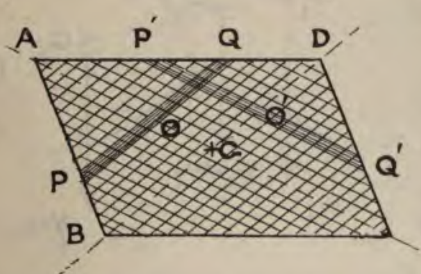
$$Q = \frac{HE}{DE} \times \frac{CG}{CH} W,$$

$$R = \frac{DH}{DE} \times \frac{CG}{CH} W.$$

Let W = 3·6 pounds.

6. A rectangular platform ABCD, whose C.G. is at G is to be supported by props at A, B, and one other point. Given $AB = 6$ feet, $BC = 12$ feet, $AG = BG = 5$ feet. Find the position of the third support if the pressures on all three are equal.

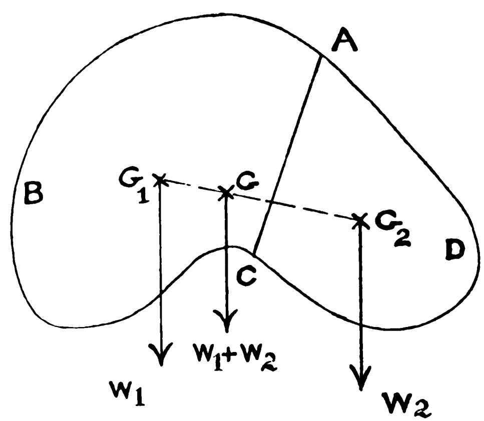
Rectangle and Parallelogram.—A rectangle or parallelogram may be split up into strips parallel to one pair of sides. The line joining the middle points of this pair of sides bisects all the strips. The C.G. will therefore lie in



this straight line. It will also lie in the straight line which bisects the other pair of sides of the parallelogram. The point of intersection of these lines gives the C.G. It obviously coincides with the intersection of the diagonals, for each diagonal is the locus of the centres of straight lines parallel to the other diagonal.

It may be noticed in these cases that every line through the C.G. is bisected at the C.G.

C.G. of Composite Body.—The position of the C.G. of a body may frequently be obtained by dividing the body into simpler parts, the positions of the C.G. of the several parts being known. For this purpose the following problem is useful. The positions of the C.G. and the weights of the two parts into which a body is divided are known, it is required to find the C.G. of the whole body.



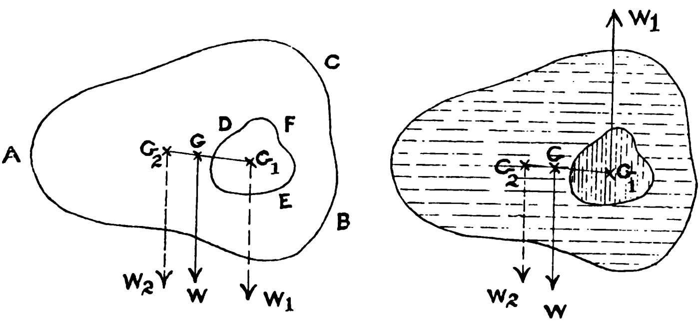
Let the figure represent a vertical section of the body through G_1, the C.G. of the part of which ACB is the section and G_2 the C.G. of the part of which ACD is the section. Let W_1 and W_2 be the weights of these parts. Then the point required will be the centre of two like parallel forces, W_1 and W_2, acting at G_1 and G_2, and therefore is a point dividing G_1G_2 in the ratio W_2 to

W_1. The C.G. of the body therefore lies in the line joining G_1 to G_2, and divides it at G, so that $\frac{GG_1}{GG_2} = \frac{W_2}{W_1}$.

Example.

7. If a straight line is drawn through the C.G. of a plane area, will it necessarily bisect the area?

C.G. of a Remainder.—It is frequently necessary to determine the position of the C.G. of the part of a body which remains when a given part has been cut away. The result just proved enables us to do this, when the weight and position of the C.G. of the original body and of the part cut off are known.



Suppose the part DEF of weight W_1 is cut out of the body ABC of weight W. Let G be the C.G. of the whole body ABCD and G_1, the C.G. of the part which is cut out, then if G_2 is the C.G. of the remaining part, we know that G_2 must lie in the line joining G_1 and G because, as we have proved, the resultant of W_1 at G_1 and W

at G_2 (where $W_2 = W - W_1$) is W acting at G, where $W_2 \times G_2G = W_1 \times G_1G$.

In the present case we really have to find the resultant of two unlike parallel forces $W_1 + W_2$ and W_1, or, as is often said, we have to find the C.G. of weights W at G and $-W_1$ at G_1.

EXPERIMENT 2.

Take an ordinary wooden ruler graduated in inches and tenths, and place on it a penny. Balance the ruler, with the penny on it, across a pencil lying on a table (see

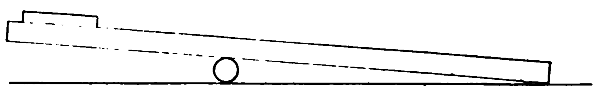


figure). The pencil should be at right angles to the ruler.

Record the distance of the opposite edges of the penny, and of the point about which the ruler balances, from the end of the ruler. Move the penny to a different position and repeat the observations. From your results construct a table, as follows—

Distance of C.G. of penny from end of ruler.	Distance of C.G. of ruler and penny from end of ruler.

Where is the C.G. of the ruler? Show the above results in a graph. How far does the C.G. of the ruler

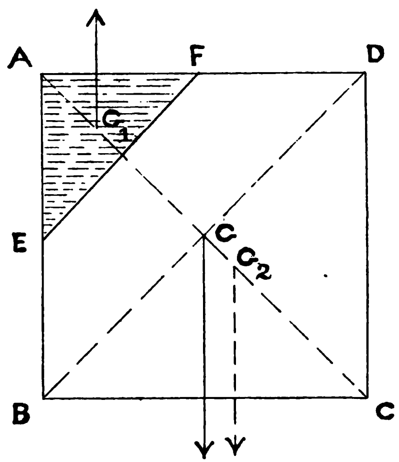
and penny move when the penny is shifted one inch along the ruler?

Can you infer the weight of the ruler, knowing that a penny newly minted weighs one-third of an ounce?

If the penny may have lost 5 per cent. of its weight in wear, what is the error introduced in the weight of the ruler, when found by assuming that the penny is of full weight?

EXAMPLES.

8. Write directions for finding the weight of a small object with only a pencil, a ruler, and a penny as appliances.



9. One of the corners of a square, ABCD, each side of which is 1 foot, is cut off across the line EF, the points E and F being the middle points of AB and AD. Find the C.G. of the remaining part.

If we denote the weight of 1 square inch of the plate by w, the whole weight of the plate is $144w$ and the weight of the part cut off $= \frac{1}{8} \times 144w = 18w$. We have therefore to find the C.G. of weights $144w$ acting at G the centre of the square, and of $-18w$ acting at G_1, the C.G. of the triangle.

Since the C.G. of both triangle and square lie on the diagonal AC, the C.G. of the part remaining also lies on AC. Let G_2 be the required C.G. Then taking moments about the centre of the square—

$$GG_2(144w - 18w) = GG_1 \times 18w,$$

$$GG_2(126w) = \frac{2}{3} AG \times 18w,$$

$$\therefore \quad GG_2 = \frac{2}{21} \times AG.$$

But $$AG = \frac{12}{\sqrt{2}},$$

$$\therefore \quad GG_2 = \frac{4\sqrt{2}}{7} = \cdot 81 \text{ inches}.$$

EXAMPLES.

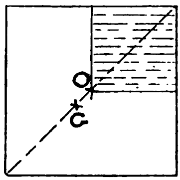
10. In each of the opposite figures find the C.G. of the unshaded part. State the length of OG.

In I. and II. O is the centre of a square, each side of which is 1 foot long.

III. and IV. are equilateral triangles of height 1 foot. In III. O is the middle point of the height, and in IV. O divides the height in the ratio of 1 : 2.

The shaded triangle in V. has its height and base each

equal to half the diagonal of the square. The side of the square is 1 foot.



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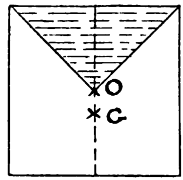

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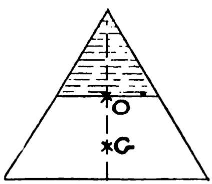

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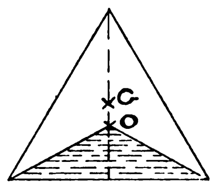
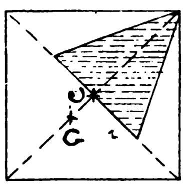
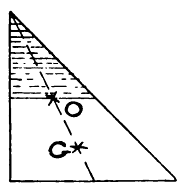


IV



V



VI

VI. is half a square, of height 1 foot, and O is the middle point of the median.

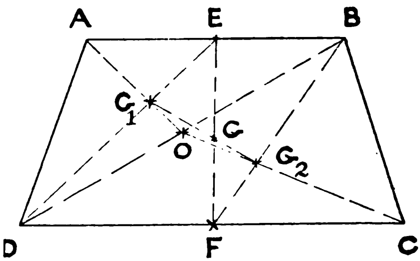
Examples.

11. A circular plate has an equilateral triangular hole cut in it, such that one angular point is at the centre of the circle and the other two on the circumference: find the position of the C.G. of the remainder. What would be the result of increasing the angle at the centre?

12. From a regular hexagonal plate of uniform thickness one of the equilateral triangles with its vertex at the centre and a side for a base is cut away. Find the C.G. of the remainder.

13. From the corner of a square card whose side is 6 inches another square is cut away whose side is 2 inches. Find the C.G. of the remaining piece.

Trapezium.—If we can find two straight lines, each of which passes through the C.G. of the body the position of the C.G. is known. Thus to find graphically the C.G. of a trapezium let ABCD be the trapezium, having AB

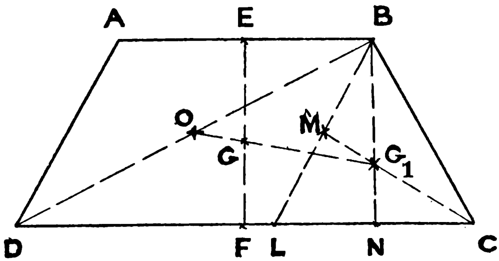


and CD parallel sides, bisect AB in E, BD in O, and DC in F. Join AO and DE meeting in G_1, then G_1 is the C.G. of the triangle ADB, similarly G_2 is the C.G. of the triangle BDC. Therefore the C.G. of the trapezium lies somewhere in the line G_1G_2. But if we divide the trapezium into strips parallel to AB and CD the centres

of all these strips will all lie in the line EF, and therefore the C.G. of the whole trapezium will also lie in the line EF. The intersection of EF and G_1G_2 is the C.G. of the trapezium.

Examples.

14. As an alternative position for the line G_1G_2 in the above construction, divide the trapezium into a parallelogram and a triangle (see figure below).



15. The parallel sides of a trapezium are 3 and 5 inches long and the line joining their middle points is 4 inches long. Find by construction and measurement the distance of the C.G. of the trapezium from the middle point of the parallel sides.

16. Show how to construct the C.G. of any plane quadrilateral by the above method.

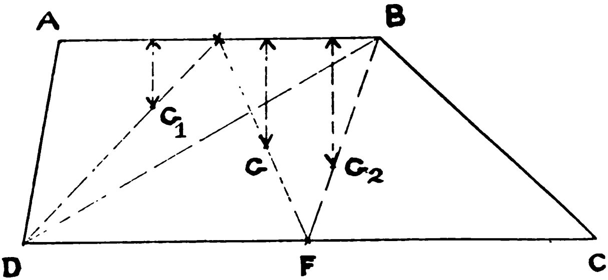
C.G. of Trapezium by Calculation.—Denote the lengths of the parallel sides AB and CD by a and b and their distance apart by h.

The two triangles into which the diagonal BD divides the trapezium are in the ratio $a : b$ because they have the same height. The distances of the C.G. of G_1 and G_2 of these triangles from AB are $\frac{1}{3}h$ and $\frac{2}{3}h$, and therefore

taking moments about AB, if z denotes the distance of the C.G. of the trapezium from AB then—

$$z(a+b) = \frac{h}{3}a + \frac{2h}{3}b,$$

$$\text{and } \therefore z = \frac{a+2b}{3(a+b)}h,$$



Similarly the distance from CD $= h - z$,

$$= \frac{2a+b}{3(a+b)}h.$$

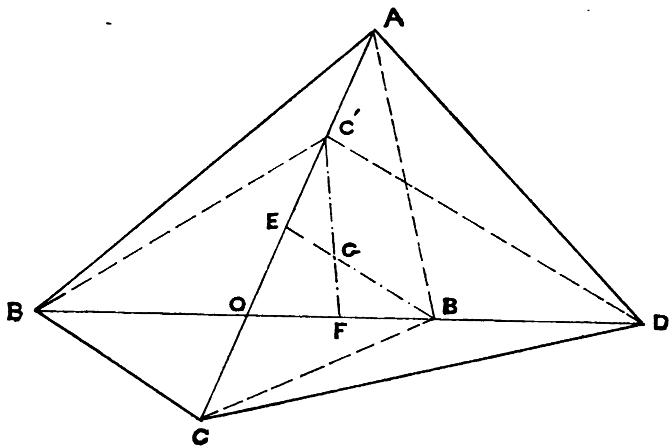
Since the C.G. lies in the medial line EF, it will therefore divide EF in the ratio $a + 2b : b + 2a$.

Example.

17. Prove the following construction for the C.G. of a trapezium ABCD, whose sides AB and CD are parallel. Bisect AB and CD in E and F. Produce AB to K making BK = DC and produce CD to L making DL = AB. The intersection of LK and EF is the C.G.

C.G. of any Quadrilateral by Construction.—Let the diagonals meet in O. Let the weight of the triangle ABD be m, and of the triangle BCD be n, and let us suppose the former to be the greater. We can replace the

triangle ABD by three equal weights of $\frac{m}{3}$ at the angles and triangle BCD by three equal weights $\frac{n}{3}$ at the angles. The C.G. of the quadrilateral is therefore the same as the C.G. of weights $\frac{m}{3}$ at A, $\frac{m+n}{3}$ at B, $\frac{m+n}{3}$ at D and $\frac{n}{3}$ at C. Since however the triangles ABD and BCD have



a common base their areas and therefore their weights are proportional to their altitudes, that is to AO and OC. That is to say, AO : OC :: $m : n$. If we cut off AC′ = OC then weights $\frac{m}{3}$ at A and $\frac{n}{3}$ at C are equivalent to weight $\frac{m+n}{3}$ at C′. The C.G. of the quadrilateral is therefore the same as the C.G. of three weights each equal

to $\frac{m+n}{3}$ at B, D and C′, that is to say, the C.G. of the quadrilateral coincides with the C.G. of the triangle BDC′.

The C.G. of the quadrilateral may now be easily constructed as follows.

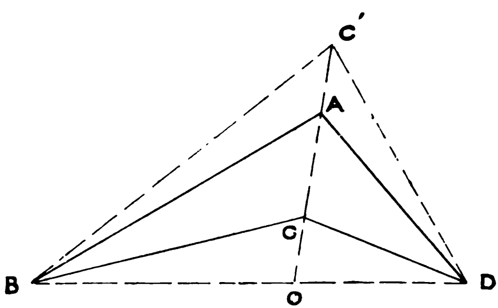
We can show in a similar way that the C.G. coincides with that of the triangle ABC′ where DB′ = BO.

Bisect BD in F, and AC in E, then the intersection of C′F and B′E gives the C.G. of the quadrilateral.

EXAMPLES.

18. In a quadrilateral ABCD, AB = 3·9 cms., BC = 5·2 cms., CD = 6·0 cms., DA = 2·5 cms., and AC = 6·5 cms. Construct graphically the C.G. of the quadrilateral and measure its perpendicular distances from AB and BC.

19. Find graphically the C.G. of a re-entrant quadrilateral. Show that if ABCD is such a quadrilateral, and if AC and BD meet in O,



and AC′ is made equal to OC, the C.G. will coincide with the C.G. of the triangle BC′D.

20. A heavy re-entrant quadrilateral plate ABCD has the following dimensions. AB = AD = 10 inches, BC = 9 inches, CD = 7 inches, and AC = 4 inches. If it is supported horizontally at A, B and D find graphically how the weight is distributed between the three supports.

In certain re-entrant quadrilaterals it will be found that the C.G. does not lie in the plate itself, but lies in the triangular space BCD. The position is however none the less definite, and it is fixed relatively to the quadrilateral itself. A hollow sphere is another case in which the C.G. does not lie in the material.

EXAMPLES.

21. A re-entrant quadrilateral is cut out of a piece of tin foil. How would you test whether the C.G. lay inside or outside the material of the plate without actually constructing its position?

22. The C.G. of a uniform quadrilateral plate is in one of its diagonals. Show that this diagonal bisects the other diagonal.

23. The diagonals AC and BD of the quadrilateral ABCD intersect in O, and the triangle ADC is twice the $\triangle$ ABC. Prove that if E and F are the middle points of AC and OD, the C.G. of the quadrilateral will be at G in EF, where EG $= \frac{1}{3}$EF.

C.G. of Several Weights.—If we have a number of small weights scattered over a plane, the position of their C.G. can be determined by referring them to two rectangular axes in the plane.

Let the magnitudes of the small weights be denoted by $w_1, w_2, w_3, \ldots w_n$, and let the co-ordinates of the points on the plane at which they are placed referred to two rectangular axes be (x_1y_1), (x_2y_2), (x_3y_3) . . . (x_ny_n). Denote the co-ordinates of the C.G. by $(\bar{x}\ \bar{y})$. Place the plane yOx vertical in such a manner that Ox is horizontal. Then the system of vertical forces w_1, w_2, w_3 has a resultant $w_1 + w_2 + w_3 \ldots$ acting at G, and the sum of the moments of the individual weights about any point O

is equal to the moment of their resultant about O,

or $w_1x_1 + w_2x_2 + w_3x_3 + \dots w_nx_n$

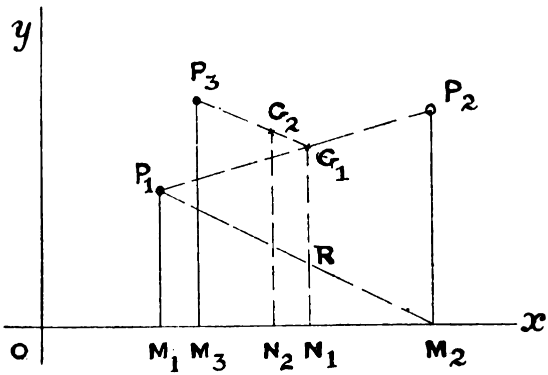
$$= \bar{x}(w_1 + w_2 + w_3 + \dots w_n),$$

or
$$\bar{x} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \dots w_nx_n}{w_1 + w_2 + w_3 + \dots w_n}.$$

Similarly, if we place the plane vertical with Oy horizontal, and write down the equation of moments about O, we get

$$\bar{y} = \frac{w_1 y_1 + w_2y_2 + \dots w_ny_n}{w_1 + w_2 + \dots w_n}.$$

Note: We could have arrived at these results by geometrical considerations.



The weights w_1 and w_2 acting at P_1 and P_2 may be replaced by w_1 and w_2 acting at G_1 where $\frac{w_1}{w_2} = \frac{G_1P_2}{P_1G_1}$. Denote the co-ordinates of G_1 by $(\bar{x}_1 \bar{y}_1)$, then join P_1M_2 meeting G_1N_2, in R.

From similar triangles P_1G_1R and $P_1P_2M_2$—

$$\frac{G_1R}{y_2} = \frac{w_2}{w_1 + w_2},$$

and from similar triangles M_2RN_1 and $M_2P_1M_1$—

$$\frac{RN_1}{y_1} = \frac{w_1}{w_1 + w_2},$$

and $$\bar{y}_1 = G_1R + RN_1 = \frac{w_1y_1 + w_2y}{w_1 + w_2}.$$

So we may prove $$\bar{x}_1 = \frac{w_1x_1 + w_2x_2}{w_1 + w_2}.$$

Similarly, we may replace $w_1 + w_2$ at G_1 and w_3 at P_3 by $w_1 + w_2 + w_3$ at G_2, where $\dfrac{P_3G_2}{G_2G_1} = \dfrac{w_1 + w_2}{w_3}$; and if we denote the co-ordinates of G_2 by $(\bar{x}_2\bar{y}_2)$ we can show in the same way that—

$$\bar{y}_2 = \frac{(w_1 + w_2)\dfrac{w_1y_1 + w_2y_2}{w_1 + w_2} + w_3y_3}{w_1 + w_2 + w_3},$$

$$= \frac{w_1y_1 + w_2y_2 + w_3y_3}{w_1 + w_2 + w_3},$$

and $$\bar{x}_2 = \frac{w_1x_1 + w_2x_2 + w_3x_3}{w_1 + w_2 + w_3}$$

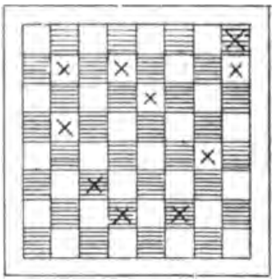
There is no difficulty in seeing that these results can be extended to the case of n weights.

Note: If the weights w_1, w_2, w_3, etc., are not small, the formulæ still hold if $(x_1\ y_1)$, $(x_2\ y_2)$, etc., are now the co-ordinates of the C.G.s of the weights, these C.G.s being all in the same plane.

Examples.

24. The centre of gravity of a triangle having its angular points at $(x_1\ y_1)$ $(x_2\ y_2)$ and $(x_3\ y_3)$ is $\left\{\frac{1}{3}(x_1+x_2+x_3)\ .\ \frac{1}{3}(y_1+y_2+y_3)\right\}$

25. Show that the triangle which has its angular points at $(-4, -6)$ $(-1, 4)$ $(5, 2)$ has its C.G. at the origin.



26. A uniform chess board weighs 25 grams. Draughts-men, each weighing 2·5 grams, are placed with their centres on the squares indicated by crosses. Find the position of the point about which the board will balance. The side of each square is 4 cms. long.

27. Find the distance from A of the C.G. of four particles of weights 3 lbs., 2 lbs., 1 lb., and 2 lbs., placed respectively at the corners of a square ABCD whose area is 32 square feet.

28. A letter T is cut out of a sheet of metal; the height of the T is 6·83 inches, the breadth of the upright piece 1·9 inches, the length of the cross piece 7·55 inches, and the breadth 2 inches. Find the C.G. Give in inches its distance from the foot of the letter.

29. If ABCDEF be a regular hexagon, whose sides are each 2 feet long, find the distance from A of the centre of gravity of the five-sided figure BFEDC.

30. Find the C.G. of the figure usually employed for proving the theorem of Pythagoras (Euclid i. 47), taking the sides of the right-angled triangle to be 3, 4 and 5 inches in length. [Hint: take as axes of reference the two sides of the right-angled triangle which contain the right angle.]

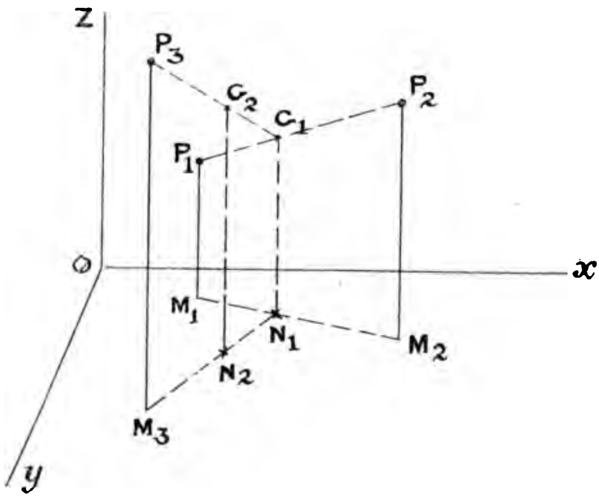
A result similar to that just proved holds when the small particles or weights are not in the same plane. If we refer the points at which the weights are placed to three axes mutually at right angles we have—

$$\bar{x} = \frac{\Sigma (w_1 x_1)}{\Sigma (w_1)},$$

$$\bar{y} = \frac{\Sigma (w_1 y_1)}{\Sigma (w_1)},$$

$$\bar{z} = \frac{\Sigma (w_1 z_1)}{\Sigma (w_1)}.$$

This result can be proved either by equating the sum ' the moments of the weights about each axis to the oment of their resultant, or geometrically in exactly the

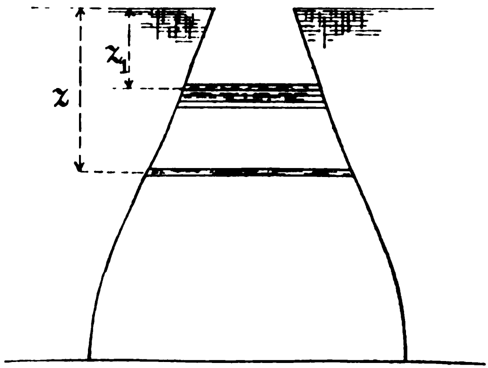


me way as for particles in one plane. The figure is 'awn for three weights, and the student ought to have › difficulty in supplying the necessary proof.

The work done in emptying a well of water by pumping e water, or by drawing the water by buckets, to the level the top is equal to product of the whole weight of water

in the well and the depth of the C.G. of the water *in situ* below the top of the well.

If we divide the water into n thin layers of weights,



$w_1, w_2, w_3, \ldots w_n$, where n is very large, and if we denote by $z_1, z_2, z_3, \ldots$ and z_n, the depths of the C.G.s of these thin layers below the top of the well, the whole work done is—

$$w_1z_1 + w_2z_2 + w_3z_3 + w_4z_4 + \ldots w_nz_n,$$

which, by the theorem just proved—

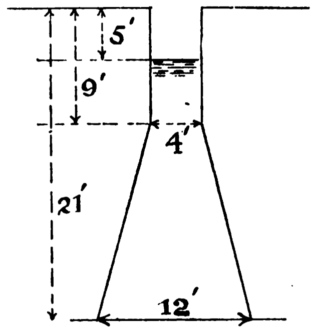
$$= (w_1 + w_2 + w_3 + \ldots w_n)\bar{z},$$
$$= W\bar{z},$$

where $\bar{z}$ is the depth of the C.G. of the well below the top, and W the weight of the water raised.

EXAMPLE.

31. Find the work done in emptying a well of water 40 feet deep the surface of the water being 10 feet below the top of the well. Suppose the well to have the form of a circular cylinder of diameter 8 feet.

Draw a diagram showing the work done at any stage by plotting on squared paper the depth of the surface of the water below the top on the well horizontally and the work done vertically.

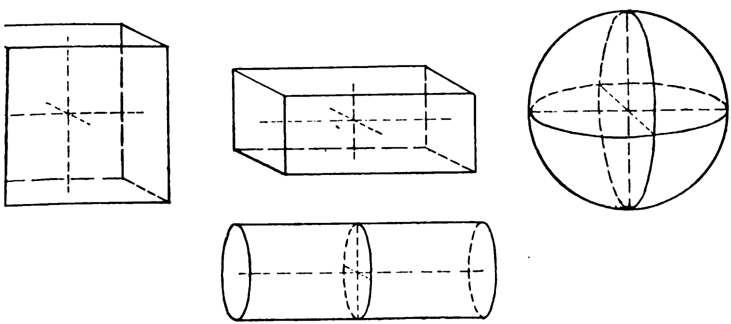


The diagram shows a well in the form of a frustum of a cone with a cylindrical neck. Find the work done in drawing the water to the top. Plot a graph showing the work done at any stage.

[Volume of frustum of cone $= \frac{\pi h}{3}(R^2 + r^2 + Rr)$.]

C.G. of Solids.—We next consider bodies of finite thickness—so called solid bodies, and we shall find that for them graphic methods cannot be so readily employed.

In the case of the simpler symmetrical solids the

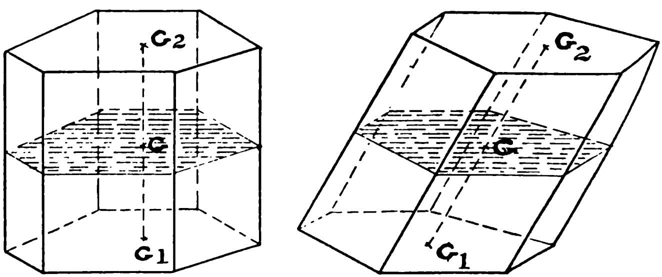


geometrical centre is the C.G. This is the case for the ube, rectangular brick, right circular cylinder of any

height, and sphere. In each of these cases every straight line drawn through the C.G. and terminated by the surface of the solid is bisected at the C.G.

As in the case of flat plane bodies it may be shown that if a solid body is suspended freely by a string attached to some point the prolongation of the string will pass through the C.G. If then we suspend the body from two different points we have two lines, each of which passes through the C.G.

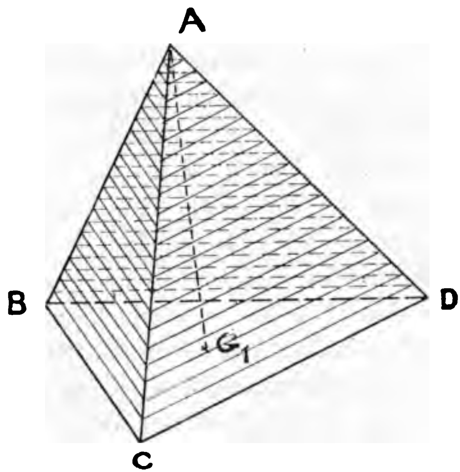
Prism.—In the case of a *prism* all sections parallel to the two parallel ends are congruent with these parallel ends.

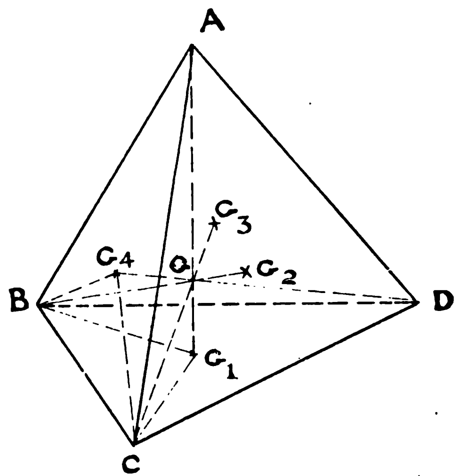


If then we divide the prism into thin slices parallel to the two parallel ends, the C.G.s of these slices will all lie in the straight line joining the C.G. of the base with the C.G. of the top. It is obvious that the C.G. of the prism lies in this line and bisects it.

Tetrahedron.—If we divide the tetrahedron into thin laminæ parallel to one face BCD, all these laminæ are similar and their C.G.s all lie in the line joining the vertex

A to the C.G. of the base BCD. Hence the C.G. of the





tetrahedron lies in the line AG_1, similarly it lies in BG_2

where G_2 is the C.G. of the face ACD. Likewise also in CG_3 and DG_4 where G_3 and G_4 are the C.G.s of the faces ABD and ABC. Hence the four lines, AG_1, BG_2, CG_3 and DG_4, all meet in one point, which is the C.G. of the tetrahedron. We have seen that G_1B, G_1C and G_1D divide the triangle BCD into three equal triangles, and therefore the small tetrahedra having G as vertex and these triangles as bases are all equal. Thus taking also the other three faces the whole tetrahedron is divided into twelve small tetrahedra, each having one vertex at G. It may be shown that these twelve tetrahedra are all equal, for the tetrahedron having A as vertex and BG_1C as base is $\frac{1}{3}$ of the whole, and the tetrahedron having D as vertex and BCG_4 as base is also $\frac{1}{3}$ of the whole tetrahedron. Denoting the volume of each of the three tetrahedra forming the tetrahedron GBCD by V_1, and similarly the others by V_2, V_3, V_4—we have $G_4BCD = \frac{1}{3}$ whole $= ABCG_1 = ABCG + GG_1BC = 3V_4 + V_1$

$$3V_1 + V_4 = 3V_4 + V_1,$$
$$V_1 = V_4,$$
and $$3V_1 = 3V_4 = (\text{in like manner})\, 3V_2 = 3V_3,$$
$$\therefore \text{tetrahedron } GBCD = \tfrac{1}{4} \text{ of the whole},$$
$$\therefore G_1G = \tfrac{1}{4} \text{ of } AG_1,$$

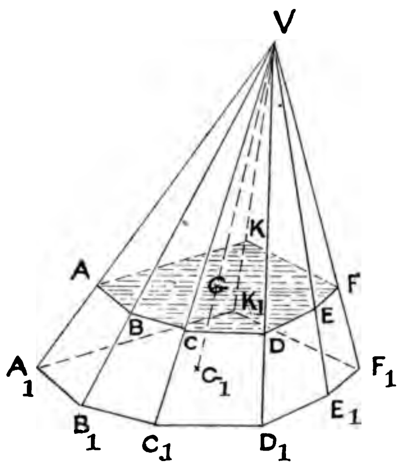
therefore the C.G. lies in the line joining A to the C.G. of the base BCD, and is three times as far from A as it is from G_1, the C.G. of the base BCD.

So we infer that the distance of the C.G. of a uniform *solid* tetrahedron from any face is *one quarter* of the distance of the opposite vertex from that face.

Example.

32. Re-draw the figures showing edges and points in the four faces in different colours, one for each face.

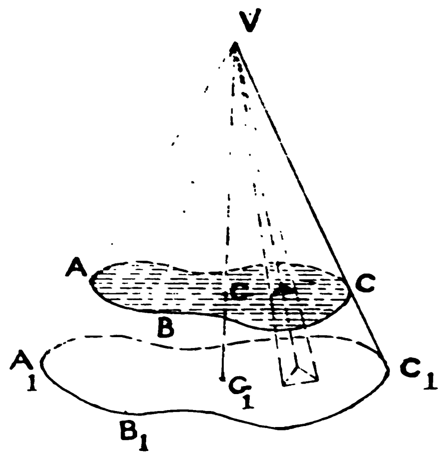
Pyramid on Polygonal Base.—Any pyramid having for base a rectilinear polygon can be built up of triangular pyramids or tetrahedra with a common vertex. If $V . A_1B_1C_1D_1E_1F_1$ is such a pyramid we draw a plane ABCDEF the distance of which from the base is $\frac{1}{4}$ of the



distance of the vertex from the base. The C.G. of each of these tetrahedra lies in the plane ABCDEF, and therefore the C.G. of the whole pyramid is also in this plane. The C.G. of the pyramid, however, also lies in the line joining the vertex to the C.G. of the base. (Why?) We have therefore the following construction for the C.G. Find

G_1, the C.G. of the base, join VG_1, and make G_1G =
Then G is the required C.G.

Any Pyramid.—An exactly similar construction gi
C.G. of a solid pyramid when the base is any closed
for we may suppose such a pyramid to consist of



large number of pyramids on small triangular bases l
a common vertex.

EXAMPLES.

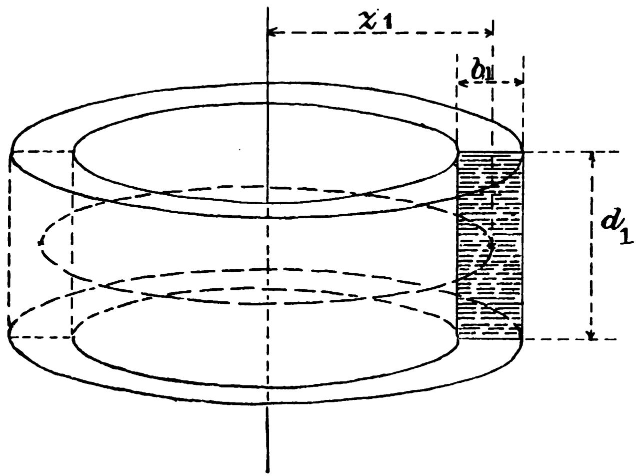
33. Find the position of the C.G. of a frustum of a cone of
h, the radii of the ends being R and r.

34. A uniform spar 60 feet long, tapers regularly from a d
of 18 inches to a diameter of 12 inches. Find the positio
C.G.

Theorems of Guldin.—If a rectangle (shaded i
figure) is rotated about a straight line in its plane p

to two opposite sides, the solid ring so formed is a hollow circular cylinder like a piece of indiarubber tubing.

Let the sides of the rectangle be b_1, d_1, and let the distance of its centre from the axis of revolution be z_1.



Volume of ring = difference between two cylinders,

$$= \pi\left(z_1 + \frac{b_1}{2}\right)^2 \times d_1 - \pi\left(z_1 - \frac{b_1}{2}\right)^2 \times d_1,$$

$$= \pi d_1\left[z_1^2 + b_1 z_1 \times \frac{b_1^2}{4} - z_1^2 + b_1 z_1 - \frac{b_1^2}{4}\right],$$

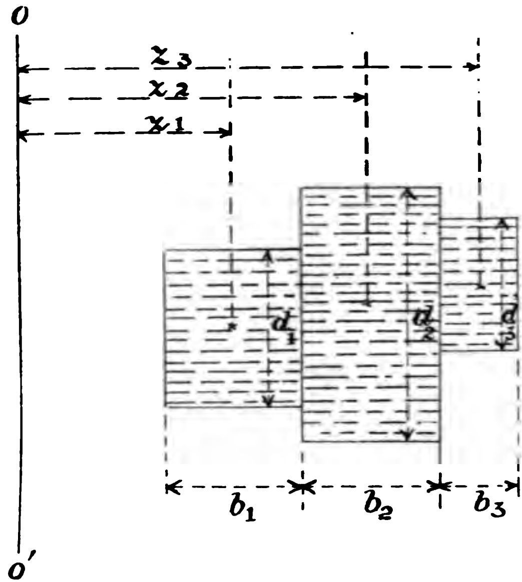
$$= 2\pi d_1 b_1 z_1,$$

$$= d_1 b_1 \times 2\pi z_1$$

Also $2\pi z_1$ is the circumference of the circle traced by the centre of the rectangle, and $d_1 b_1$ is the area of the rectangle—

$\therefore$ volume of the ring $= \begin{cases} \text{area of rectangle} \times \text{length of the} \\ \text{path of the centre of rectangle.} \end{cases}$

If we have a large number of rectangles each having



two sides parallel to OO′ the volume of the solid form by rotating them about OO′ is—

$$2\pi[b_1d_1z_1 + b_2d_2z_2 + b_3d_3z_3 + \ . \ . \ . \ b_nd_nz_n],$$

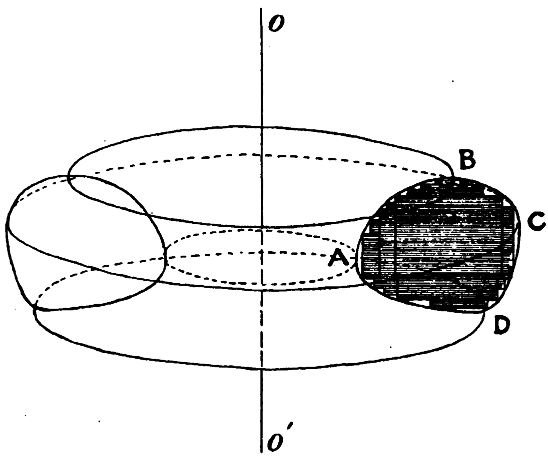
the notation employed being sufficiently explained by t figure. But if $\bar{z}$ denotes the distance of the C.G. of all t rectangles from the axis of revolution we have alrea proved that—

$$\bar{z} = \frac{b_1 d_1 z_1 + b_2 d_2 z_2 + \ldots b_n d_n z_n}{b_1 d_1 + b_2 d_2 + \ldots b_n d_n},$$

$$\therefore \text{volume} = 2\pi\bar{z}\,[b_1 d_1 + b_2 d_2 + b_3 d_3 + \ldots b_n d_n],$$
$$= 2\pi\bar{z}\,\text{A}.$$

where A is the sum of the areas of the rectangles,

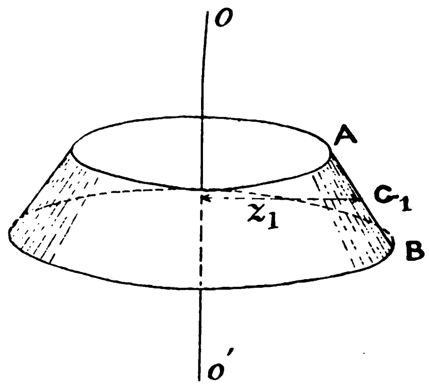
$$\therefore \text{volume of the solid} = \begin{cases}\text{length of circular path of C.G.} \times \\ \text{sum of the areas of rectangles.}\end{cases}$$



The result just stated holds if, instead of a number of rectangles side by side, we have a plane area ABCD which forms a solid of revolution by revolving about an axis in its plane. For we can fill up the whole of the area with rectangles having their sides parallel and perpendicular to the axis. We can make the sum of these rectangles differ from the area of the curve by as small a quantity as we please by filling in the small triangular pieces wit

smaller and smaller rectangles. Since the proposition holds for the sum of all the rectangles, it will hold for the area of the curve itself. Therefore—

I. "The volume of any solid formed by revolving a plane area about a line in its plane is equal to product of the area and the circumference of the circle traced out by the C.G. of the area."



If a straight line AB revolves about an axis OO′ in its own plane it sweeps out a frustum of a cone. The area of the curved surface of this frustum is $\pi(R + r)$ AB, when R and r are the radii of the ends of the frustum. Calling AB l_1, and the distance of the C.G. of the line AB from OO′ z_1, we have—

$$z_1 = \frac{R + r}{2},$$

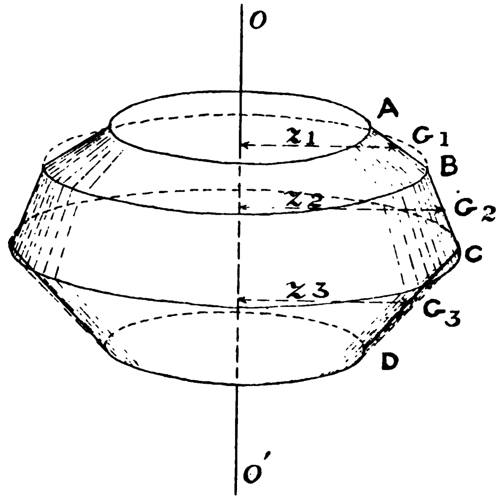
and hence the surface of frustum $= 2\pi z_1 l_1$.

So the surface formed by revolving a broken line ABCDE about an axis in its plane $= 2\pi(z_1 l_1 + z_2 l_2 + z_3 l_3 + \ldots)$,

and if we denote by L the length of the broken line and by $\bar{z}$ the distance of its C.G. from O′O, we have—

$$\bar{z}\,\mathrm{L} = z_1 l_1 + z_2 l_2 + z_3 l_3 + \ldots,$$

$\therefore$ surface of solid figure (not including the ends) $= 2\pi\bar{z}\,\mathrm{L}$.

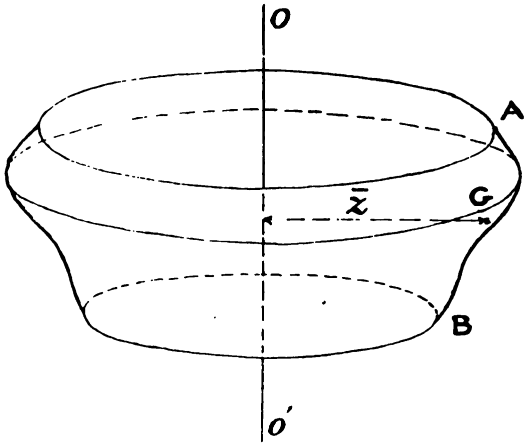


Lastly, if we take a curved line and revolve it about some straight line in its plane the proposition will still hold; for the curve may be supposed to consist of an indefinitely large number of indefinitely small straight portions (See figure, p. 226.)

The surface of any solid formed by revolving a curve about a straight line in its plane is equal to the product of the length of the curve and the circumference of the circle traced out by the C.G. of the curve.

These theorems have been known for many centuries.

They were first enunciated by Pappus, who lived about 380 A.D., a well-known geometer of the famous school of Alexandria. Guldin, a Jesuit (1577–1643), restated the theorems, but failed to give a general proof. The subject was considered about the same time by several contem-



poraries of Guldin, among whom were Kepler, the well-known astronomer, and Cavalieri, the professor of mathematics at Bologna. It seems that the credit of neither the discovery nor the proof is due to Guldin, though the theorems are universally known by his name.

Surface of a Sphere.—At a time in the world's history when the science of arithmetic can hardly be said to have existed, Archimedes (287–212 B.C.), the celebrated sage of Syracuse—who is said to have invented such wonderful engines of war, to the discomfiture of the Roman generals—showed that the circumference of a circle of unit

diameter lay between $3\frac{1}{7}$ and $3\frac{10}{71}$ units. This remarkable result was obtained by calculating the perimeters of two regular 96-sided polygons, one circumscribed to the circle, and the other inscribed in it. The former of these values is the well-known approximation to π in general use.

We propose to find in a somewhat similar way the area of the surface of a sphere, though the proof given is not so rigorous from a scientific point of view as that of Archimedes, as we shall be content to take a regular *circumscribed* polygon.

Draw a regular polygon with an *even* number of sides to circumscribe a circle. In the figure, p. 228, we have taken one of eighteen sides. AJ is a diameter of the circle, which passes through two opposite angular points, and since the number of sides in half the polygon is odd, one of the sides FE will be parallel to AJ. Draw perpendiculars on AJ through the angular points of the polygon.

If the figure is rotated about AJ the circle will trace out a sphere, and the polygon will trace out a solid consisting of a number of frusta of cones, each of which has its vertex in AJ, except of course the side EF, which will trace out a cylinder touching the sphere.

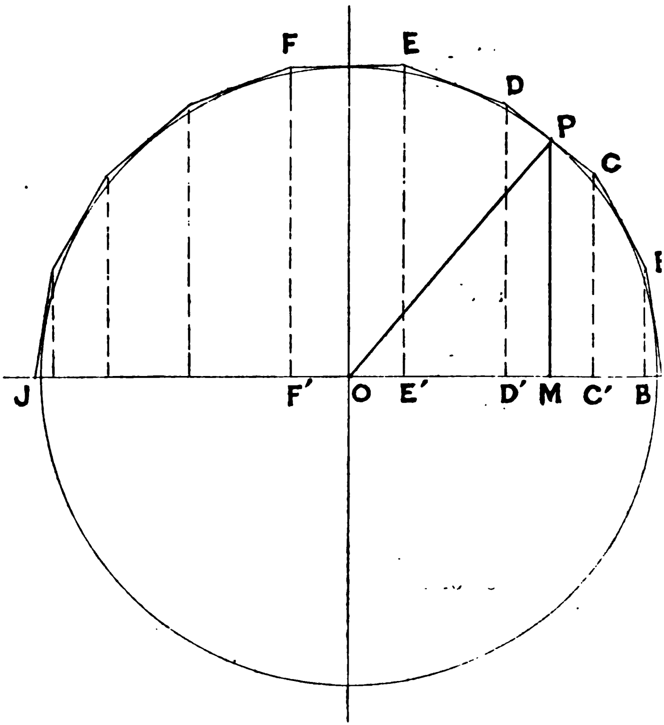
Let P be the point in which any side such as CD touches the circle, then P is the middle point of CD, and the frustum traced out by CD has its area $= 2\pi \times \text{PM} \times \text{DC}$, where PM is the perpendicular on AJ. But CD makes the same angle with D′C′ as PM makes with OP

therefore
$$\frac{\text{DC}}{\text{D}'\text{C}'} = \frac{\text{OP}}{\text{PM}}$$

and
$$\text{PM} \times \text{DC} = \text{OP} \times \text{D}'\text{C}'.$$

Therefore the area of the surface of this frustu $= 2\pi D'C' \times R$, denoting by R the radius of the sphere.

Similarly the surfaces of the other frusta can

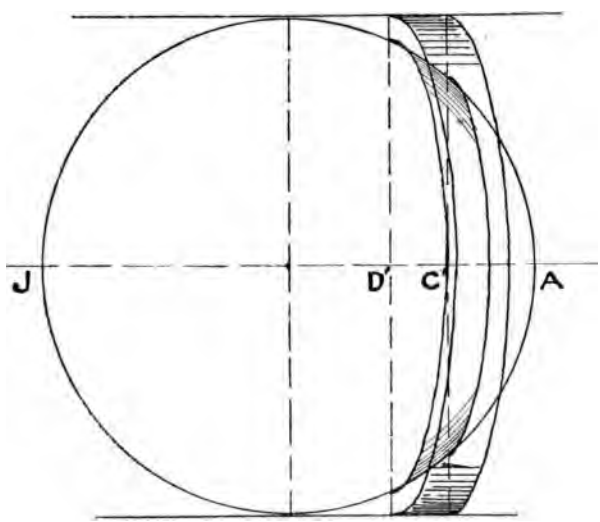


obtained, and the whole surface of the solid formed fro the polygon is—

$$2\pi R(AB' + B'C' + C'D' + \ldots I'J) = 2\pi R \times AJ.$$

If we increase the number of sides of the polygon tl diagonal of the polygon AJ will become very nearly equ to the diameter of the circle, and the solid figure form

by revolving the polygon will closely coincide with the sphere. We therefore infer that the surface of the sphere itself is equal to $2\pi R \times 2R = 4\pi R^2$. We infer also that the surface of any belt of the sphere included between two parallel planes is equal to $2\pi R \times D'C'$, where $D'C'$ is the distance between the parallel planes. If we draw a cylinder to circumscribe the sphere so as to have AJ as axis, the surface of the cylinder included between the two parallel planes will also be $2\pi R \times D'C'$, and therefore the parallel planes cut off equal surface areas from the sphere and the cylinder.



EXAMPLE.

35. Find the area on the surface of the earth included between the 50° and 51° parallels of latitude taking the earth to be a sphere of radius 3960 miles.

C.G. of Belt of Spherical Shell.—Since parallel planes cut off equal surface areas from a sphere and its circumscribing cylinder, the parallel planes being perpendicular to the axis of the cylinder, it follows that the *distribution of weight* along the axis of the cylinder is the same for the cylinder and the spherical shell. It is therefore evident that the C.G. of such a belt on a sphere coincides with the C.G. of the belt of the circumscribing cylinder. Since the C.G. of the belt of the cylinder is mid-way between the parallel plane ends, the C.G. of the belt of the spherical shell is also mid-way between the parallel planes, and as a special case, the C.G. of a hemispherical shell bisects the radius which is the axis of the shell.

EXAMPLE.

36. A hemispherical shell is suspended freely from a point in the rim, find the inclination of the plane of the rim to the vertical.

Each of the theorems of Guldin establishes a relation between three quantities: thus, the first theorem gives us an equation connecting (1) a plane area; (2) the distance of its C.G. from a line in its plane; and (3) the volume obtained by revolving it about that line. If we happen to know any two of these quantities the theorem gives us the third.

C.G. of a Semicircular Wire.—The C.G. of a wire in the form of a semicircle is immediately obtained from the surface of a sphere by the second theorem of Guldin. If we rotate the semicircular wire about the line joining the ends, we get a spherical shell, and therefore

if $\bar{z}$ is the distance of the C.G. of the wire from the centre of the spherical shell—

$$\pi R \times 2\pi\bar{z} = 4\pi R^2,$$

$$\therefore \quad \bar{z} = \frac{2}{\pi}R = \frac{2R}{\pi}.$$

EXAMPLES.

37. Find the C.G. of a wire 6 inches long bent into the arc of a circle such that it subtends an angle of 60° at the centre of this circle.

38. A semicircular wire is suspended from one end. Find the inclination of the line joining the two ends to the vertical.

39. Find the C.G. of the arc of a circle which is greater than a semicircle. [Hint: Regard the arc as part of a whole circular hoop with a piece missing.

40. Find the C.G. of the arc of a circle which subtend an angle of α radians at the centre of the circle.

[Distance from centre of circle $= \dfrac{2R\sin\frac{\alpha}{2}}{\alpha}$.]

41. Find the C.G. of a coil of wire, 1 foot in diameter, consisting of $6\frac{1}{2}$ turns.

[Distance of C.G. from centre $= \dfrac{12}{13\pi}$ of an inch.]

C.G. of a semicircle.—The first theorem of Guldin gives us the position of the C.G. of a semicircle. The volume obtained by revolving the semicircle about its diameter is a sphere. Denoting the distance of the C.G of the semicircle from the centre by $\bar{z}$ we have—

$$2\pi\bar{z} \times \frac{\pi R^2}{2} = \frac{4}{3}\pi R^3,$$

$$\bar{z} = \frac{4R}{3\pi}.$$

42. Find the position of the C.G. of a semicircle of diameter 6 inches.

Cut a semicircle, diameter 6 inches, out of cardboard, and mark its C.G., using the result already obtained. Verify by constructing its position experimentally in the manner explained at p. 191.

Examples.

43. Find the volume of the ring formed by revolving a circle of radius R about a straight line in its plane at a distance z from the centre of the circle.

44. Find the volumes of (1) a right circular cone, (2) a right circular cylinder by employing Guldin's Theorem I.

45. A *sector* of a circle is bounded by the arc and by two radii. Show that its C.G. coincides with that of a certain concentric arc.

Volume of a Sphere.—We may regard a sphere as made up of a large number of small cones, each of which has its vertex at the centre of the sphere. The volume of any one of these cones is equal to $\frac{1}{3}$ of the product of the area of the base and its height. If the base of each cone is very small, the height is practically equal to the radius of the sphere. The volume of the whole sphere is then equal to $\frac{1}{3}$ of the product of the area of the surface, and the radius $= \frac{1}{3} \times 4\pi R^2 \times R = \frac{4}{3}\pi R^3$.

C.G. of Solid Hemisphere.—In finding the volume of a sphere we regarded it as made up of a large number of small cones, each having its vertex at the centre of the sphere. In a similar way we can find the position of the C.G. of a solid hemisphere by regarding it as made up of a very large number of equal small cones, each having its vertex at the centre of the base of the hemisphere. The C.G. of each of these cones lies on a hemispherical cap or shell of radius $\frac{3}{4}$ of the radius of the

hemisphere. The C.G. of the solid hemisphere will therefore coincide with the C.G. of this hemispherical cap. The distance of the C.G. of the hemisphere from the centre of its base is therefore $\frac{1}{2} \times \frac{3}{4}R = \frac{3}{8}R$.

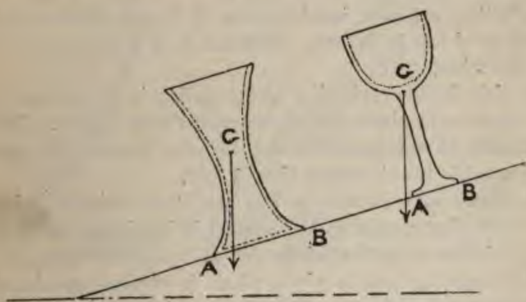
Examples.

46. A solid hemisphere is suspended from a point on the rim of the base. Find the inclination of the base to the vertical.

47. Show that the C.G. of a solid sector of a sphere coincides with the C.G. of a certain spherical cap.

Upsetting.—When a body is placed on a plane it will upset if the vertical through the C.G. falls outside the base.

For, in the diagram, the weight of the wine-glass has a



moment tending to rotate the wine-glass round A, and as it will not adhere to the plane, or, in other words, as the plane cannot exert a *pull* on it, but only a push, there can be no moment, due to the reaction of the plane to balance the moment of the weight, and so the wine-glass will upset.

The flower vase on the same inclined plane will not upset, for the moment of the weight *is* balanced by the moment due to the reaction of the plane.

Examples.

48. A prismatic block has its ends square, 1 inch side, two of its sides rectangles, and the two remaining sides parallelograms of angle 60°. Find the greatest length of the sides if it will just not upset when placed on a horizontal plane.

49. A cylinder, radius r, height h, stands on a horizontal plane, which is gradually tilted up. At what inclination will the cylinder upset, assuming that it will not slip?

50. A brick is placed on an equal brick, which projects over the edge of a table, the ends of the bricks being parallel to the table edge. Find the position of the bricks when the upper brick projects beyond the table-edge as far as possible.

51. Extend the preceding example to the case of three bricks.

52. A rectangular iron plate is bent into an arc of a cylinder, and placed upright on a circular edge. Will it stand or fall? Modify the proposition of the preceding paragraph so as to include cases in which the base has a re-entrant angle.

53. A hollow cylinder, inside radius $5\frac{1}{2}$ inches, stands on a circular table, radius of top 18 inches. How much of it can project over the edge of the table.

54. A thin hollow cylinder whose radius is 35 inches is placed upright upon a circular table whose radius is 37 inches. Prove that it will remain at rest, provided the distance between its centre and that of the table is not greater than 1 foot.

55. A square table stands on four legs at the middle points of its sides. Show that a man will not upset it by sitting on a corner if his weight is less than that of the table.

56. A triangular lamina whose plane is vertical, stands with one side on a horizontal plane, and is on the point of toppling over. If one angle of the lamina is 119° 12′, find the other two angles.

Stability and Instability.—In popular language an object is said to be stable when it is not easily displaced or moved. This is not, however, the usual mathematical sense of the word. The mathematical test of stability is, that if the body is slightly displaced and then left to itself, it comes back to its original position.

Thus, a pith ball, suspended by a silk fibre, is a type of

instability in the popular sense, for the least breath moves it, but it is mathematically stable, for if slightly displaced and left to itself it will come to rest again in its original position.

If a body is hung up by an inextensible string from a fixed point it is impossible to move the body in any manner which does not raise its C.G. The body is in stable equilibrium. On the other hand, imagine a weight

fixed to the top of a stick, which can turn about its lower end. Every movement of the weight lowers the C.G. The equilibrium is unstable.

These obvious facts suggest the statements: "If the C.G. of a body is in the lowest possible position, its equilibrium is stable; if the C.G. is in the highest possible position the equilibrium is unstable."

A position may be stable for one kind of displacement and unstable for another. For instance, a rock placed at

the summit of a "col" or saddle-shaped depression in a range of hills.

A position may exist in which the equilibrium is *neutral.* In this case a small displacement does not alter the height of the C.G. For example, a sphere lying on a horizontal table.

EXAMPLES.

57. Point out positions of stable, unstable, and neutral equilibrium for a cone resting on a horizontal table.

58. A round table of radius R feet has a vertical leg with a three claw foot. If the claws are at distances D feet apart, find the condition that no weight placed on the table will upset it. As a particular case let $R = 1$ foot.

59. It has been proposed to adopt as an axiom, "The C.G. of any body or system of bodies will descend as far as it can." Consider three equal rods, AC, CD, BD, each 3 inches long, hinged at C and D and suspended from A and B. Replace the weight W of each rod by $\frac{1}{2}$W at either end. We see that the C.G. of the three rods coincides with that of $\frac{1}{2}$W at A, $\frac{1}{2}$W at B and 2W at O the middle point of CD. Therefore, adopting the axiom, we see that in the position of equilibrium O must be as low as possible. Find the position of O, and hence the equilibrium position of the rods by trial, if A is $1\frac{1}{2}$ inches above B and 5 inches to the left of it.

60. A and B are points in the same horizontal line, 2 feet apart. AB, BC, CD are three strings knotted together, of lengths 28, 16, and 30 inches, and weights of 3 lbs. and 5 lbs. are attached at B and C. Apply the axiom of the preceding example to find the position of equilibrium by trial.

A more general mode of considering the question of stability is as follows: Imagine the body slightly displaced from its position of equilibrium. The forces acting on it in its displaced position will not in general be in equi-

librium. If their resultant tends to bring the body back to its original position the equilibrium is stable, for that particular displacement. If their resultant tends to carry the body still further from its original position the equilibrium is unstable.

EXAMPLES.

61. A weight W is suspended by two strings which pass over two smooth pulleys and have weights each of amount P attached to them. Is the position of equilibrium stable for vertical displacement? Examine the case when $P = \frac{1}{2}W$.

62. A chain hangs over a smooth pulley, the ends being in the same horizontal line. Is the equilibrium stable or unstable?

If a body can turn about an axis there will be two positions of equilibrium, one stable and one unstable. In both positions of equilibrium the C.G. will be in the vertical plane containing the axis, but in the stable position the C.G. will be below, and in the unstable position it will be above, the axis.

It is sometimes arranged that a bucket or wagon is movable about an axis which is *below* its C.G. when it is full, so that it may be easily tipped over. To keep it from turning over when full, it will have to be locked or clamped in some manner. When the clamp has been unfastened a slight push only is required to make it tip over. The position of the axis may be so arranged that the wagon or bucket is stable when empty and unstable when full.

Thus in the case of a conical bucket, if the axis is situated between a $\frac{1}{4}$ and $\frac{1}{3}$ of the height it will be stable when empty and probably unstable when full, unless it is filled with some very light material.

In the case of the wagon figured in the diagram let us suppose that it is 10 feet long and 5 feet broad at the top and that it is 6 feet deep, and hence $6\frac{1}{2}$ feet slant side.

If z_1 is the depth of the C.G. below the top, then—

$$z_1 = \frac{10 \times 6{\cdot}5 \times 3 + 5 \times 3 \times 2}{10 \times 6{\cdot}5 + 5 \times 3} = \frac{225}{80}$$
$$= 2{\cdot}81 \text{ feet from top.}$$

Supposing that the wagon is made of iron plates $\frac{1}{4}$ inch thick, weighing 10 lbs. per square foot, then the total weight is $20(10 \times 6{\cdot}5 + 5 \times 3) = 1600$ lbs. If, however, it is full of earth weighing 110 lbs. per cubic foot the total weight of the earth $= 15 \times 10 \times 110 = 16500$ lbs., and the C.G. of the earth will be 2 feet from the top. The distance of the C.G. of the whole from the top will then be—

$$\frac{16500 \times 2 + 1600 \times \frac{225}{80}}{16500 + 1600} = 2{\cdot}07 \text{ feet from top.}$$

If then the axis of rotation is situated at a distance of say $2\frac{1}{2}$ feet from the top the wagon will be stable when empty but unstable when full of earth.

Examples.

63. A heavy uniform triangular plate rests on three horizontal supports at the angular points, prove that, whatever its shape may be, the pressures on the three supports are equal.

64. A round table has three legs situated at equal distances 7 feet

apart near the rim. If a load of 100 lbs. is placed on the table at a distance of 5 feet from each of two legs, find the increase in pressure of each leg on the floor.

65. A lamina has the shape of a square with equilateral triangles described externally on two of its adjacent sides. Show that the distance of its C.G. from the centre of the square is $\frac{\sqrt{3}-1}{4} \times$ the length of the diagonal of the square.

66. A circular lamina of 12 inches radius is punched with a circular hole of one inch radius whose centre is the middle point of a radius of the lamina. Find the C.G. of the remainder.

If a second equal punch is made at the centre of the lamina, find the C.G. of the remainder.

67. Particles of weights 1, 2, 3, 4, 5, 6 lbs. are placed at the corners A, B, C, D, E, F respectively of a regular hexagon ABCDEF. Find the distance of their C.G. from the point A in terms of the side AB.

68. Weights of 1 lb., 2 lbs., 3 lbs., 4 lbs. are placed at the corners A, B, C, D respectively of a square ABCD whose sides are 10 inches long. Find the distance of their C.G. from the corner D.

69. A pendulum consists of a bar 30 inches long and weighing 4 ounces and a bob at the end of the bar weighing 15 ounces. How far is the C.G. of the pendulum from the bob?

70. A letter L is cut out of a rectangular sheet of metal, measuring 28·4 inches by 22·3, by the removal of a rectangle 20·2 inches by 14·1 inches, so that the breadth of each limb of the L is 8·2 inches. Show that the L can be suspended by a point in the highest edge so as to hang vertically.

71. A heavy uniform straight rod, 12 inches in length, has to balance on its end. To effect this, from the centre of the rod and opposite sides of it, side pieces of the same material and thickness are fixed in downward directions each making an angle of 60° with the vertical. Find the least length of the side pieces for an effective balance.

72. A rectanglar piece of cardboard, 6 inches by 8 inches, has a circular hole of diameter 3 inches cut from it. If the centre of this hole lies on a diagonal at a distance of 3 inches from one corner, find the position of the point about which it will balance.

73. A rectangular table, 5 feet by 4 feet, and weighing 100 lbs., is hung by three vertical cords, one attached at the middle of a 4-foot

side, the other two at the ends of the opposite side. Find the pull on each.

74. 50 tons is moved through a distance of 22 feet transversely across the deck of a ship. The weight of the ship, including this, is 3,000 tons. In consequence of this movement the C.G. of the total weight must undergo a displacement relatively to the vessel. Calculate the magnitude and direction of this displacement.

75. A sheet of iron 3 feet square is divided into squares of 1 foot by lines parallel to the sides; one of these is removed from the middle of one side; find the C.G. of the remainder, giving its distance from two adjacent sides.

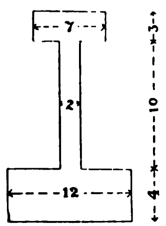
76. A regular polygon ABCD . . . with n sides is circumscribed to a circle whose centre is O, and whose radius is $(n-1)$ feet. How far from O is the C.G. of the figure formed by the polygon with the triangle OAB removed?

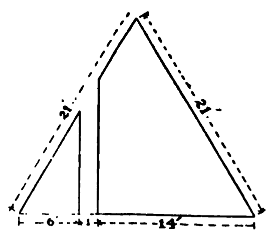
77. If O is the centre of a circle ABC and OB = 2 feet, BOC = 30°, find the C.G. of the figure formed by the circle ABC with the triangle BOC removed.

78. A heavy circular disc, weighing 10 lbs., rests on three horizontal pegs near the rim, the angles subtended at the centre of the disc by the lines joining the pegs being 90°, 130° and 140°. Find by graphical construction and measurement the pressure on each peg.

79. A circular hoop hanging vertically is suspended from a smooth hinge about which it can revolve freely in its own plane. A weight equal to that of the hoop is attached to the extremity of the diameter perpendicular to that drawn from the hinge: find the position of equilibrium.

80. Calculate the position of the C.G. of the area in the figure, the measurements given being in inches.





81. The sail plan for a small boat is given (not to scale) in the accompanying diagram. Find the centre of the sail area.

82. A fly-wheel, weighing 1250 lbs., has its C.G. $\frac{1}{20}$ inch from its centre. To make the C.G. coincide with the centre a hole 2 inches in diameter is drilled in the wheel. Where ought this hole to be made if the wheel is of steel $2\frac{1}{2}$ inches thick and weighing 0·28 lb. per cubic inch?

83. A fly-wheel is a disc 14 inches in diameter. The C.G. is 0·07 inch out of the centre. The defect is to be remedied by drilling a hole in the heavy side of the wheel 5 inches from the centre of the wheel. To what diameter should this hole be drilled?

Balancing Machinery.—The preceding examples serve to illustrate a point of great importance in machinery, viz. that each separate revolving part should have its C.G. in its axis of rotation. If the C.G. of a fly-wheel is not in its axis of rotation the wheel is said to be badly *balanced.* Any want of "balance" in the revolving parts of an engine produces wear of the bearings, and sets up vibrations in all parts of the engine. This causes a loss of energy, and may even be a source of danger.

Fly-wheels of the disc pattern, as employed on the common road locomotive, are balanced by being machine-planed all over, so as to bring the C.G. truly central.

The carriage-wheels of the London and North-Western Railway are balanced experimentally. To test whether the two wheels on one axle are truly balanced, the axle with the wheels on it is placed on bearings mounted on springs, and the wheels are rotated at about 450 revolutions per minute. Any want of balance is immediately detected by the vibrations of the springs. Plates are attached to the wheel till it runs steadily without vibration.[1]

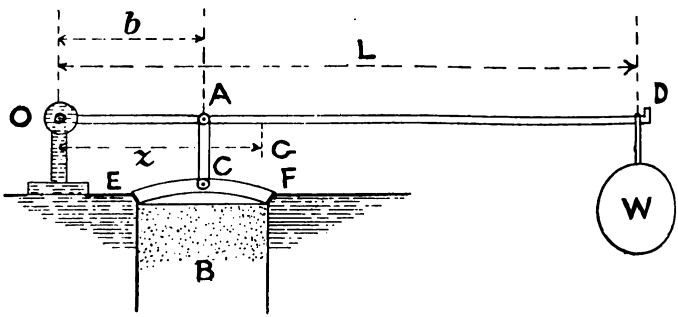
[1] *Balancing of Engines*, p. 20, by Prof. W. E. Dalby.

EXAMPLE.

84. Supposing that a plate of 1 lb. attached at 1·2-foot radius balances a wheel weighing 969 lbs., find the original deviation of the C.G. from the axis.[1]

[0·000124 foot measured from the axis on the line joining the centre of the balancing plate to the axis produced.]

Safety Valve.—In order that the pressure of the steam in a boiler should not become so great as to cause an explosion, all boilers are fitted with safety valves which allow the steam to escape when its pressure exceeds a certain amount. These safety valves take many different forms. One of the simplest forms is the lever safety valve (see figure). In this type the lever arm OD can



move about O as a fulcrum, and carries a weight W at the end D. The seat EF of the valve is attached by a link AC to the lever arm. When the steam pressure in B becomes so great as to lift the seat of the valve from its bearings the steam will blow off. This will occur when the moment of the steam pressure about O exceeds the

[1] *It must not* be thought that the only requisite for balancing is *that the C.G.* should be in the axis. This is not so.

moment of the weight W, together with the moment of the weights of the lever arm, link, and seat of the valve about O.

The pressure produced by a liquid or a gas on any surface in contact with it is everywhere at right angles to the surface. The pressure on a plane area is uniform if the forces on all equal small portions of the area are the same. The pressure is usually estimated in lbs. per square inch. In the case of a uniform pressure on a plane area, if p is the pressure in lbs. per square inch and A is the area in square inches, then the resultant pressure or total force on the area is pA lbs. In this case the resultant pressure acts at the C.G. of the area, because the effect would be the same as if the area were horizontal and a large number of equal small weights placed on it close together so that the weight on any square inch should be p lbs.

Let G be the C.G. of the lever arm together with the link and seat, and z the horizontal distance of G from O,

l = distance of the centre of the valve from O,

L = length of the lever arm,

p = limiting pressure of steam in lbs. per square inch,

d = diameter of valve in square inches.

The steam will just be on the point of blowing off when—

$$p \times \frac{\pi d^2}{4} \times l = \mathrm{WL} + wz,$$

or

$$p = \frac{4(\mathrm{WL} + wz)}{\pi d^2 l}.$$

85. A lever safety valve has the following dimensions—mean diameter, $2\frac{1}{4}$ inches; weight of valve, 7 lbs.; distance of centre of valve from fulcrum, $2\frac{1}{4}$ inches; weight of lever, 14 lbs.; distance of C.G. of lever from fulcrum, 12 inches. Where must a weight of 20 lbs. be hung from the lever so that the valve may lift at a pressure 100 lbs. per square inch above atmospheric pressure?

86. A circular disc 4 inches in diameter and 1 inch thick has a semicircular groove of breadth $\frac{3}{4}$ of an inch cut round its edge. Find the volume of the disc.

87. The volume of a frustum of a cone of height h having R and r as the radii of its ends is $\frac{\pi h}{3}(R^2 + r^2 + Rr)$. Find from this result the position of the C.G. of a trapezium of height h and parallel sides R and r, having two adjacent angles right angles.

CHAPTER VIII

FRAMES

Any Number of Forces acting at a Point in One Plane.—We have seen that if *three* coplanar forces are in equilibrium their lines of action must either meet in a point or be parallel. When four or more coplanar forces are in equilibrium their lines of action do not *necessarily* meet in a point. The case in which the lines of action *are* concurrent is, however, of great practical importance, and lends itself to simple treatment.

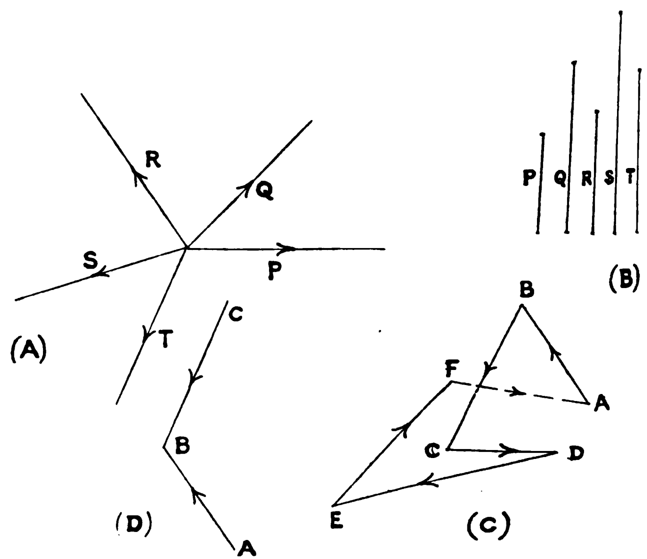
The problem we are now to consider may be stated as follows—

A number of forces, given in magnitude and direction, act at a point in one plane, it is required to find their resultant graphically. For the sake of obtaining a definite diagram we consider the case of five forces, all of which act outwards from the point of application, but the reasoning may be applied to any number of forces acting inwards or outwards.

Let forces of magnitude P, Q, R, S, T act along the lines shown in figure A. The relative magnitudes of the forces P, Q, R, S and T are shown by the parallel lines in *Fig. (B), p. 246.*

Draw a line AB to represent one of the forces, say R, and a line BC to represent one of the others, say T. Remember that BC has to represent the *direction* as well as the magnitude of the force T. Thus for the case we are dealing with Fig. (C) is right but Fig. (D) is wrong.

The line AC, as we have seen, represents the resultant of the forces R and T. Now draw CD to represent one of



the other forces, say P. The resultant of the two forces represented by AC and CD is represented by AD, and since AC represents the resultant of AB and BC, AD represents the resultant of the three forces R, T and P. Similarly, if we draw DE to represent S, AE will represent *the resultant* of the four forces R, T, P and S. And again

if EF is drawn to represent Q, then AF will represent the resultant of the five forces R, T, P, S and Q.

It should be observed that the forces may be taken in any order that is convenient. The essential matter is that each force should be correctly represented in the force diagram, not only in magnitude but in direction. If an arrow is put into each side of the polygon to indicate the direction of the force represented by that side, it will be noticed that if we imagine ourselves walking round the polygon, starting in the direction indicated by the first arrow, we shall throughout the circuit be always walking in the direction of the arrows.

An interesting exercise is the following: Consider five forces of 5, 6, 7, 8 and 9 lbs. acting at O in directions making angles of 0°, 60°, 150°, 210° and 330° with a fixed line OX.

Draw a line AB to represent the force of 5 lbs. There are twenty-four distinct modes of completing the diagram to furnish the resultant. If each member of a class draws one, or two, of these the whole series may be obtained.

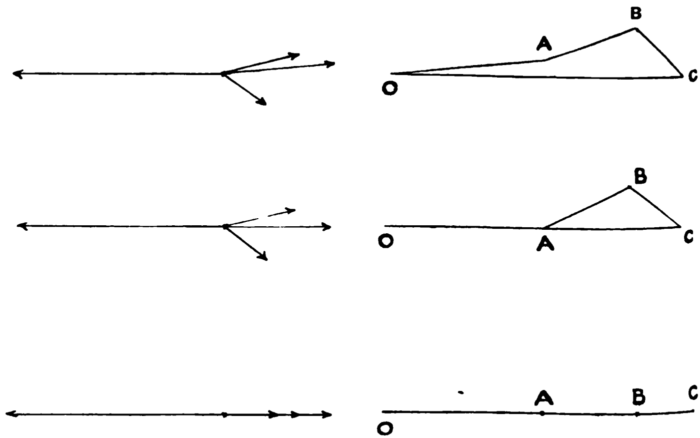
If we applied a sixth force at O, represented by FA, this force, being equal and opposite to the resultant of the other five, will be their equilibrant. This is a case of the general proposition known as the *polygon of forces*, which may be enunciated as follows—

"If any number of forces acting at a point are represented by the sides AB, BC, CD . NA of a closed polygon, they are in equilibrium."

The converse problem is: A number of coplanar forces acting at a point are in equilibrium. How are their

magnitudes and directions related? There is *a* polygon whose sides represent the forces, and the question is, what data will enable us to construct this polygon? If there are more than three forces, a knowledge of their directions is *not* sufficient. The reason is that equiangular polygons are not necessarily similar, for example, a square is not similar to a rectangle.

In other words, if a number of forces acting at a point



are in equilibrium, they are represented on some scale by the sides of a certain set of similar polygons, termed force polygons, but not by the sides of *any* polygon which can be drawn with its sides parallel to the directions of the forces. Commencing with the case of four forces, P, Q, S and T, acting at a point in given directions we see that if the magnitudes of two are known these two may be replaced by their resultant R, a known force. We now have three forces, R, S, T, acting at a point in given

directions, and can at once find the magnitude of S and T by drawing a triangle.

Hence we can draw a force polygon for the set of four forces if we know the magnitudes of two of the forces.

We may notice that if the lines of action of two or more of the forces are the same, the polygon may have two or more of its sides in a line.

In the first figure opposite the forces represented by OA and CO are nearly parallel, in the second they are actually parallel, and in the third all the four forces represented by OA, AB, BC, CO are all parallel, and the polygon reduces to a single line.

Example.

1. Five forces acting at a point in given directions are in equilibrium. How many must be known in magnitude for a force polygon to be drawn?

It will be now easily seen that however many forces acting at a point in given lines are in equilibrium, we can find the magnitudes and directions of two of the forces if the magnitudes and directions of all the rest are known. A convenient mode of doing this will be illustrated by an example.

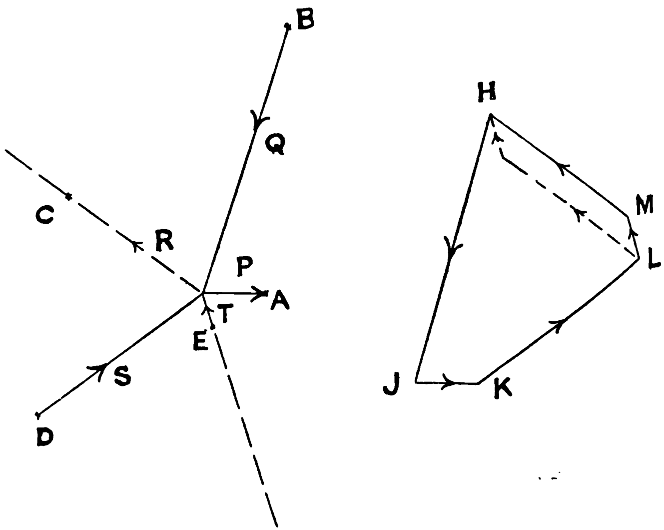
Five forces, P, Q, R, S and T, act in one plane at a point O, in lines OA, OB, OC, OD, OE inclined at equal angles (of 72°) to one another.

Given $P = 5$ lbs., $Q = 22$ lbs., and $S = 16$ lbs., and that the directions of P, Q and S are indicated by the arrows, find the magnitudes and directions of R and T.

Commencing with any one, say Q, of the three given forces, draw a line HJ to represent it.

Draw JK to represent a second, say P, of the given forces, and KL to represent the remaining one.

We notice that HL represents the resultant of P, Q and S, and therefore must represent the equilibrant of R and T; thus R and T are the components of this resultant

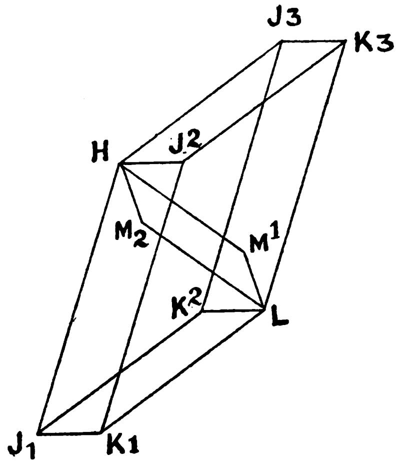


along OC and OE. To obtain these we draw a line through H parallel to one of the lines OC or OE, say OC, and a line through L parallel to the other OE. Let these lines intersect in M, then LM represents T and MH represents R.

It must be carefully observed that LM determines not

only the magnitude of T but also its direction, *i.e.* in a direction from E towards O, and not from O to E.

The student will find that there are twelve ways of drawing the figure. They are represented in the figure.



Any continuous path, HJKLMH, along the lines of the diagram determines a force polygon.

Examples.

2. Three forces, P, Q, R, of 10, 20 and 30 lbs., act at a point in the directions S., N. 65° E., N. 55° W. Find the resultant, and the angle it makes with the force Q.

3. The following forces act at a point—15 lbs. due E., 20 lbs. N.E., 40 lbs. N. 30° W., 50 lbs. S. Find the magnitude of the resultant, to the first decimal of a pound, and express graphically the direction, giving the tangent of its inclination to the E. and W. line.

4. Find the magnitude and the direction of the resultant of the four following forces acting at a point: 5 lbs. weight acting E., 4 lbs. N., 9 lbs. W., and 8 lbs. S.

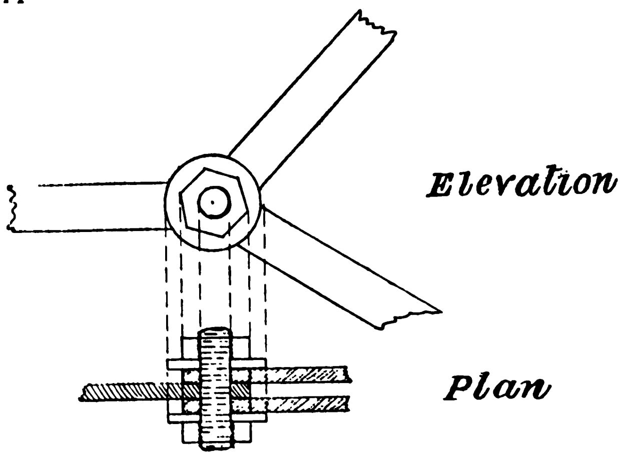
5. Three equal forces of 100 lbs. act at a point, the first in direction E., and the other two at equal intervals of 72° round by the N. By the "polygon of forces," or otherwise, show that their resultant is 162 lbs., and find its direction. What is the direction and magnitude of the resultant of four such forces?

6. Four bars, OA, OB, OC, and OD, are jointed together at O, the angles between consecutive bars are 25°, 125°, 140°, and 70°. If the *pull* in OA is 25 lbs. and the *thrust* in CO is 45 lbs. find the stresses in OB and OD which will keep the joint O at rest.

7. Four forces acting in a plane at a point O, are represent

in magnitude and direction by the lines OA, OB, OC, and OD. OA = 2 inches, OB = 3·5 inches, OC = 1·2 inches, OD = 4 inches. ∠ AOB = 15°, ∠ AOC = 90°, ∠ AOD = 221°. An inch represents 10 lbs. weight. Draw a diagram to scale to represent the forces and measure off from the diagram the value of the resultant OF and the ∠ AOF. [Army.]

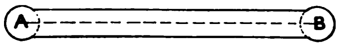
Frames.—An important type of structures, much used for roofs and bridges, are known as framed structures or frames. A "frame" is a structure composed of bars or rods jointed to one another. Two instances are shown at pp. 260 and 261.



We will suppose the joints to be "pin joints," such that the bars could turn freely about the pin. The action of the pin on the bar is, in the circumstances, equivalent to a single force (see p. 187).

Let us suppose that any weights or external forces which are supported by the frame, are applied at the pins or joints.

Consider the forces acting on a typical bar AB of such a frame, namely—



(1) The action P of the pin at A on the bar, (2) the action Q of the pin at B on the bar, (3) the weight of the bar itself.

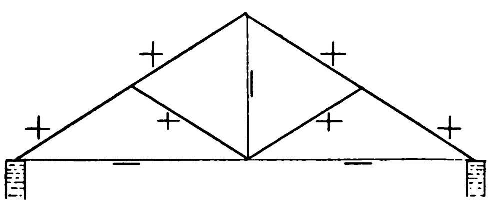
The weight of the bar itself will obviously in many cases be small compared with the loads, and we will neglect it.

The forces P and Q must therefore balance one another, and this is only possible if they are equal and they act in the line joining A and B. Hence the bar is subject either to a direct *pull* or to a direct *thrust.*

A bar in a state of tension, that is, subject to a direct pull on either end, is technically termed a **tie.**

A bar in a state of compression, that is, subject to a direct push or thrust on either end, is technically termed **a strut.**

In the diagram representing a structure (frame diagram) the struts and ties are distinguished by + and − signs, thus—



The result just proved may be enunciated thus: "If a frame is loaded at the joints, if the joints are free, and

we neglect the weight of the bars themselves, then each bar of the frame is either a strut or a tie, *i. e.* is subject only to a direct longitudinal force."

It may seem that as the assumptions on which this proposition is based are not strictly true, any conclusions deduced from it will be of small practical value, but this is by no means the case. Any stiffness or lack of freedom in the joints evidently increases the stiffness of the structure as a whole, and so the error caused by neglecting the stiffness of the joints is, *as a rule*, an error on the right side, that of safety.

Again, the effect of the weight W of any one bar on the remainder of the frame may be correctly found by replacing W by two forces each $\frac{1}{2}$W at the ends of the bar. There only remains the bending effect on the bar itself, due to its own weight. This bending effect is often though not always negligible. It is discussed fully in books on applied mechanics.

Advantages of a Framed Structure.—By experimenting on a small piece of wood, such as a wooden match, the student will realize that it is not possible to pull it in two by a fair direct pull. He may also show by placing two matches in a vertical position in small holes bored in a piece of wood, and then piling books or weights so as to rest half on the matches and half on a fixed support, that they can support a considerable thrust, or compression. On the other hand, he will find that a match supported at its two ends will only support a comparatively small weight at its centre.

If time permits the experiments may be repeated with slips of stronger woods than the soft pine generally used for matches.

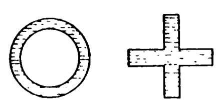
Similar experiments indicate that solid bars of wood and many metals possess considerable tensile and compressive strength, but are ill-suited to resist bending, and hence the important conclusion may be drawn—

"The bars which make up a framed structure are subjected to the kind of 'stress' or distorting force, which they are best fitted to resist."

An important difference between a bar in tension and a bar in compression must now be noticed. In the case of a tie the pulls on the two ends tend to keep the bar straight, while in the case of a strut, if the bar gets bent the forces applied to it tend to bend it more. To prevent the bar bending it must have considerable section. In a structure such as an iron roof or a girder of a bridge it is frequently possible to find out which bars are in tension and which are in compression by observing their construction.

Ties are usually solid bars of small section, either round or rectangular. On the other hand, the compression bars or struts have to resist the tendency to "buckle" or bend, and must have sections suitable for this purpose.

SECTIONS OF TIES.

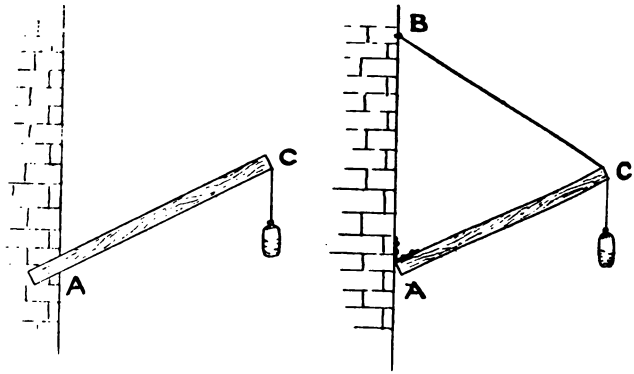


SECTIONS OF STRUTS.

The difference in construction may be noticed in the

iron roofs of railway stations and in many railway bridges. Look also at the shape of the ribs of an umbrella frame, and also at the shape of the stalk of any tall plant,[1] *e.g.* wheat, fennel, figwort, sedge.

Consider now a beam, AC, built into a wall, as shown in the figure. A weight W hung from C, evidently tends to break the beam off short at A. If we tie the



end C up to the wall by a string BC, we relieve the beam to some extent; and we may prevent the weight at C from having any tendency to bend the beam by hinging the beam to the wall at A.

We will neglect the weight of the beam. The beam AC is acted on by two forces, viz. the reaction of the wall at A and a force at C, which is the resultant of the weight W and the tension T of the string BC. These forces cannot be in equilibrium unless they have a common line of action, and this common line of action can

[1] *Brit. Assoc. Rep.*, 1904, p. 812.

only be the line CA. The direction of the resultant and also of the equilibrant of W and T, is in the line CA. Hence, if W is given, the magnitude of T and of the thrust of the beam on the end C, which thrust is the equilibrant of W and T, can be found.

Examples.

8. B is a point on a vertical wall 3 feet above A. A beam, AC, $2\frac{1}{2}$ feet long, is hinged at A, the end C being attached to B by means of a rope $2\frac{1}{2}$ feet long. Find the thrust in the beam and the tension of the rope caused by hanging a load of 84 lbs. at C. Why is it quite unnecessary in this case to draw a force diagram?

9. A weight of 10 lbs. is suspended by means of two strings, each 6 inches long, from two points A and B, in the same horizontal line. If AB has the following values, 0, 2, 4, 6, 8, 10 and 12 inches (very nearly), find the corresponding tensions of the strings. Draw on squared paper a curve showing how the tension varies as AB increases.

10. If one string was 4 inches long and the other 8 inches in the last example, make a table of values of the tensions of the two strings (now different) for the same values of AB. Draw two curves on the same piece of squared paper, showing how the tensions alter as AB increases.

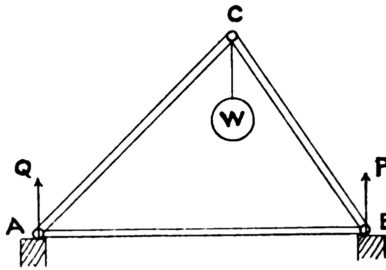
Triangular Frame.—Consider a triangular frame ABC composed of three jointed bars of given lengths. Let a given weight W be hung from C, and the frame be supported with AB horizontal by vertical reactions Q and P at A and B.

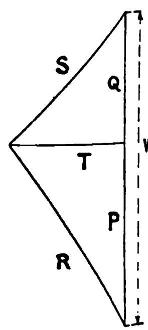
The forces acting on the joint or pin at C are W, the thrust S of the bar AC, and the thrust R of the bar BC. These forces act in known directions, and hence a force triangle may be drawn.

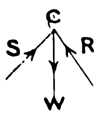
Example.

11. Let AC = 7 inches, BC = 9 inches, AB = 13 inches, and W = 5 lbs. Draw the force triangle for the joint C.

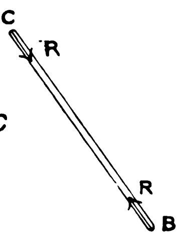
Next consider the bar BC. It is acted on by two forces, the reaction of the pin at C and the reaction of the pin at B. These, as we have already noticed, must be equal and act in the line BC. Since the action of BC on the pin at B is equal and opposite to the reaction of the pin at B on the bar BC, the action of BC on the pin at B is equal to the known force R.



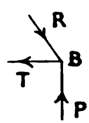




Forces on C



Forces on BC



Forces on B

The forces acting on the joint B are three in number—(1) The thrust R of the bar CB; (2) the vertical pressure P of the support; (3) the pull T of the bar AB; and the triangle of forces for the joint B can be drawn.

EXAMPLES.

12. Draw the triangle of forces for the joint B, using the data and result of Example 11.

13. Specify the forces acting on the joint A, and draw the triangle of forces for this joint, using the result of the previous example (12).

14. Draw the triangle of forces for the joint A, using the result of Example 12. Are the answers of 13 and 14 in agreement?

The student should have no difficulty in seeing that the three separate triangles of forces for the joints A, B and C can be put together, like the pieces of a puzzle, into a single figure, as shown.

The advantage of drawing this single figure instead of three separate triangles is obvious. We have only six lines to draw and measure instead of nine. As there are six distinct forces in question, it is clear that no simpler representation of their magnitudes and directions is possible than that given by the above figure.

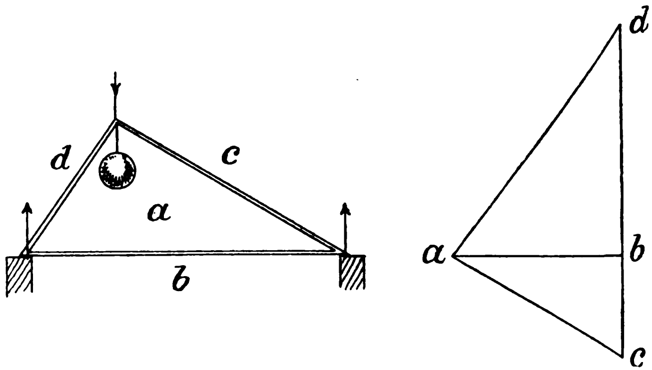
Any framed structure may be regarded as a collection of joints, each of which is kept in equilibrium by a number of forces acting in given directions.

A "force polygon" exists for each joint; and these force polygons may be combined into a single figure which may be termed its force diagram.

Bow's Method.—A method of indicating corresponding lines in the force and frame diagrams was proposed by Mr. R. H. Bow[1] (*Economics of Construction*, 1873), which,

[1] It is stated in Cotterill's *Applied Mechanics* that the method was suggested, at a meeting of the London Mathematical Society in 1871, by Professor Henrici.

though at first sight it may appear artificial, possesses striking advantages in complicated cases.



In the above figure the frame and force diagrams of p. 258 are re-drawn and lettered in accordance with Bow's notation.

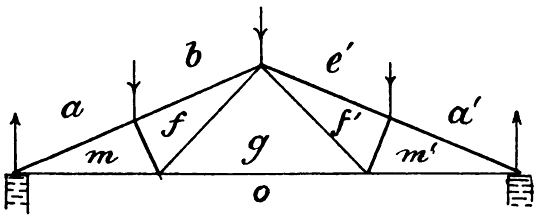
Commencing with the frame diagram we treat it as if it were a map; that is, we give distinguishing names or letters to the regions or spaces into which the lines of the figure divide the plane. A line is named by the regions between which it is the boundary. The frame diagram is lettered in the ordinary way as a geometrical figure, and it will be found possible to name the points on it in such a way that any line, for example *ad*, has the same name as the parallel line in the frame diagram. The principle of the method is therefore that corresponding parallel lines in the frame and force diagrams shall be named alike.

EXAMPLE.

15. A triangular frame ABC is acted on by forces at ABC at right angles to the opposite sides. Given AB = 11 in., BC = 12 in., CA = 13 in., and the force at A = 15 pounds acting inwards—specify the forces acting on the joint A, the joint B, and the joint C. Draw the force triangle for each joint, and put the force triangles together into one force diagram. Letter the frame diagram and the force diagram by Bow's method.

If the joint A and the force triangle for it are drawn in red, B and its triangle in black, C and its triangle in blue, the connection between the frame and force diagrams will be clearly shown.

The following example will show the method of procedure—A roof frame of iron, span 40 feet, is represented



in the figure. The short bars are at right angles to the long rafters (each 22 feet) at their middle points.

We will draw the force diagram on the supposition that the frame is loaded with W cwts. at the apex and at the centre of the long rafters, and supported by vertical reactions.

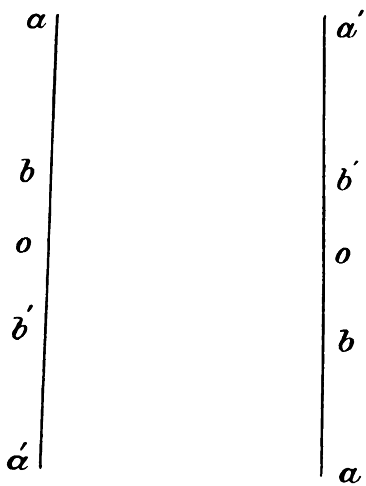
It is clear, from the fact that the frame and the loads are both symmetrical on either side of the roof, that the reactions at the points of support are equal, and each $\frac{3W}{2}$.

Step I.—Draw the frame diagram to scale, indicate the lines of action of the external forces and letter the spaces.

Remark.—We only letter the spaces in order to enable us to name each bar, and each external force. We shall avoid increasing the number of spaces unnecessarily if we never draw a line of action of an external force so as to cut across the frame, or in both directions away from the frame. Adherence to this rule will save much trouble.

Step II.—Draw the force polygon for the external forces and letter it on Bow's principle.

Remarks.—(1) The frame, as a whole, is in equilibrium under the external forces. Imagine the frame while keeping its shape to become smaller and smaller, the forces keeping their original magnitudes and directions. Evidently the frame would still be in equilibrium. The frame may be supposed to become so small that the external forces act at a point. We can therefore draw a force polygon for the system of external forces.



(2) In this case, and in most of the cases in which the frame is subject to vertical loads, this polygon degenerates into a line (cf. pp. 248, 249).

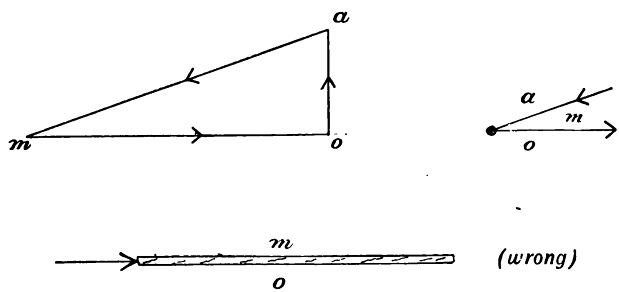
(3) There are always two ways of doing this, as shown above.

Step III.—Find a joint for which we can draw the triangle or polygon of forces. Draw this triangle or polygon and letter it. Such a joint in the present case is *amo*. The points *a* and *o* in the force diagram have been obtained. Draw *am*, *om*, parallel to the lines *am*, *om* in the frame diagram.

Remark.—(1) A convenient way of noting which points are already known and which are to be found, is to underline those already known, $\underline{a}$, m, $\underline{o}$.

Step IV.—Find whether the bars at the joint in question pull it, or push it, *i.e.* whether they are struts or ties. Here *am* pushes and is a strut, *mo* pulls and is a tie.

Remarks.—(1) This is best indicated on the frame diagram, by putting + against a strut, − against a tie.

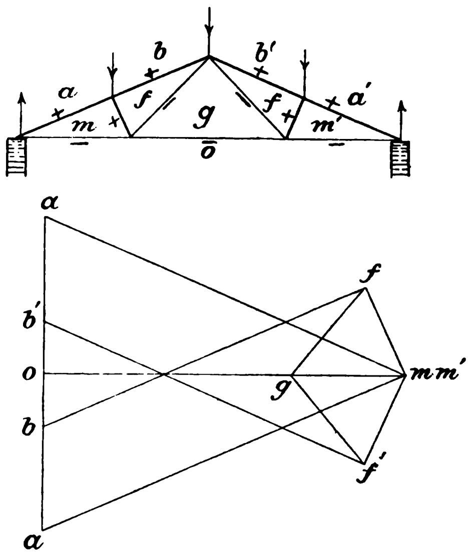


(2) The triangle *amo* shows the directions of the forces acting on the *joint*. Beware of thinking that the joint exerts a force on the bar *mo* in the direction shown in the lower figure.

Step V.—Repeat Step III. for the remaining joints in turn, namely—

(1) $a\ m\ b\ f$,	(4) $b'\ a'\ m'\ f'$
(2) $o\ m\ f\ g$,	(5) $o\ m'\ a'$,
(3) $b\ b'\ f'\ g\ f$,	(6) $o\ g\ f'\ m'$.

The final frame and force diagrams are shown below.



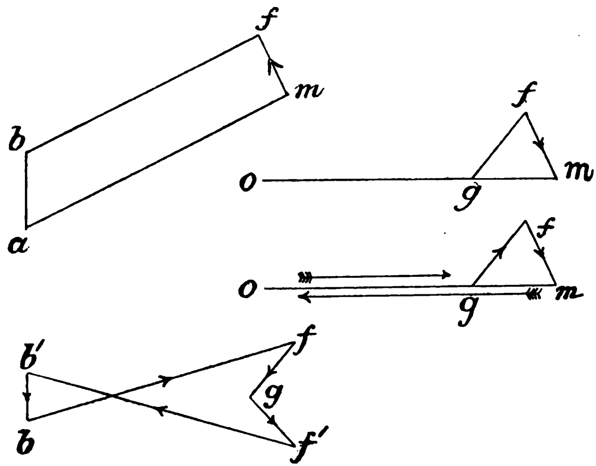
Remarks.—(1) We observe that at each step we deal with a joint for which at most only one new point has to be fixed in the force diagram.

(2) The parallelism of the last line drawn, to the corresponding bar of the frame diagram, furnishes a check on the accuracy of the work.

(3) Any line in the force diagram either represents an

external force or else it forms part of *two* of the separate force polygons for two joints.

For example, consider the line *mf*. This forms part of the polygon *abfm* and also of the polygon *mfgo*: *mf* repre-



sents a force directed upwards—the push on the joint *abmf*; *fm* represents a force directed downwards—the push on the joint *omfg*.

It is for this reason that arrow-heads should never be placed on the complete force diagram, though there is no harm in putting arrow-heads on a force polygon for a single joint, if it is drawn separately. Some writers place arrow-heads on the frame diagram, but this too is apt to lead to mistakes, and in any case conveys no further information than is given by the + and − signs.

Common-sense will often show that a particular bar of a frame is a strut, or a tie, as the case may be. Thus

it is obvious that *am* is a strut. In the case of the bar *fg*, the answer to the question "strut or tie" is not so obvious, and we now explain more fully how to find the answer from the force diagram. The bar *fg* acts on the joint at the ridge, and the polygon for this joint is *bb'f'gf*. The load on this joint is represented by *b'b*, and its direction is downwards, therefore the directions of the other forces on the joint will be indicated by going round the polygon, starting from *b'* towards *b*. Showing the course taken by arrows we see that the bar *fg* is *pulling* the joint and is therefore a *tie*.

What is the bar *fb*?

If we wish to consider the joint *omfg*, we notice that as the bar *fg* pulls on the joint at the ridge it must pull also on this joint.[1]

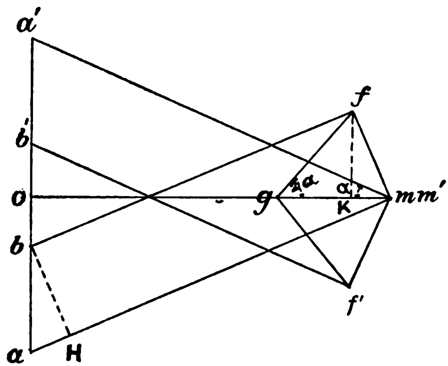
(4) The force, either thrust or tension, in a bar may be conveniently termed the *stress* in the bar.

(5) It might be supposed in a case in which both the frame and the load are symmetrical with respect to a vertical axis, that only half the Bow's diagram need be drawn. This is not the case. The important check which the completion of the Bow's diagram furnishes is lost if the diagram is left unfinished.

A table giving the magnitude of each stress with its proper sign should be drawn up. A convenient notation for the stress in the bar *fg* is S_{fg}.

[1] The student may satisfy himself by holding a pencil one end in each hand that a bar can pull the ends to which it is fastened together, or can thrust against both ends, but being an inanimate object, cannot pull one end and at the same time push the other.

Calculation from Bow's Diagram.—Knowing the inclinations of the bars of the frame to the horizontal, we can readily calculate the relative lengths of the lines in the force diagram, and thus the magnitude of the stresses represented. Thus, in the frame just dealt with, the long rafters being inclined at a to the horizontal, and the short struts at right angles to the rafters, the angle fgm is $2a$, and we find—



$$S_{am} = S_{a'm} = \frac{3W}{2 \sin a},$$

$$S_{bf} = S_{b'f'} = \frac{3W}{2 \sin a} - W \sin a,$$

$$S_{om} = S_{om'} = \frac{3W}{2} \cot a,$$

$$S_{fm} = S_{f'm'} = W \cos a, \qquad [fm = bH]$$

$$S_{fg} = S_{f'g} = \frac{S_{fm} \cos a}{\sin 2a} = \frac{W}{2} \cot a,$$

$$[fg \sin 2a = fK = fm \cos a]$$

$$S_{og} = W \cot a.$$

Design of a Tie-rod.—Suppose that, by drawing a Bow's diagram for a given frame, subject to given loads, we have obtained the following results—bar *ab* tie, tension 10 cwts.; bar *bc* tie, tension 5 cwts. The question arises what should be the thickness of these bars?

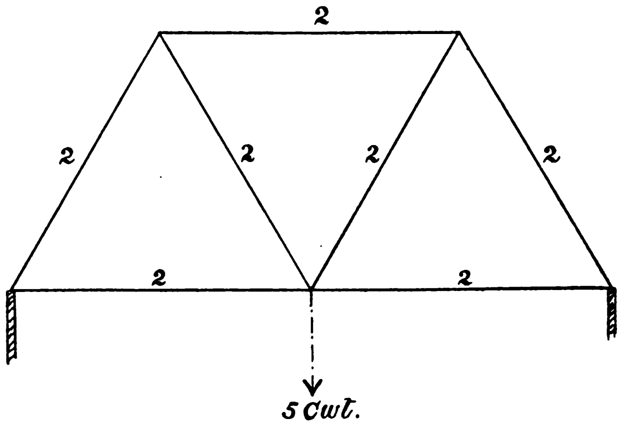
If a certain bar will support a pull of 5 cwts., it seems clear that two such bars side by side will support a pull of 10 cwts. equally distributed between them, and that if the two bars are joined together to form a single bar, this result will still hold good. We conclude therefore that the strength of two bars of the same material to resist tension is in direct proportion to the area of their cross section.

By the term "same material," we do not mean simply that the rods are both for instance of iron, but that they are of the same quality of iron, equally free from flaws and variations. The thicker the iron the greater the difficulty of ensuring uniformity of quality. The design of long struts is a more complicated matter, and cannot be discussed here.

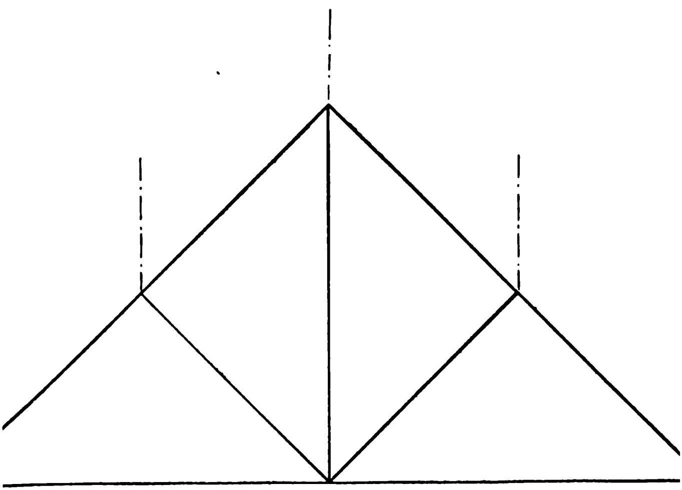
Examples.

16. What diameter should be given to a tension rod to resist a pull of $1\frac{1}{2}$ tons if the tensile stress should not exceed 4 tons per square inch?

Draw the frame diagrams of the structures shown, with their loads in the following figures. Draw the force diagram for each, and tabulate the stresses in the bars.



17. Reactions vertical ; triangles equilateral.



Span, 20 feet : slope, 45°.

18. Reactions vertical; load, 2 cwts. at apex, and at centre of each long rafter. This frame is known as the "King Post" and is often to be seen in large barns or in houses which are being built.

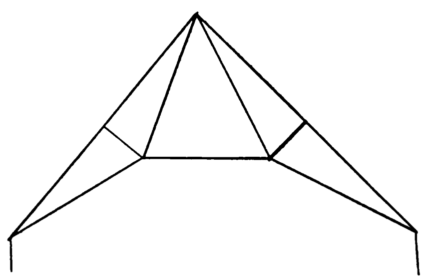
19. Same roof; unsymmetrical load, viz. 2 cwts. at apex, 1 cwt. and 2 cwt. at centres of long rafters. Assume the reactions vertical, and find them by taking moments about A and B in turn. Check by means of their sum.

20. Consider the same roof with the loads of Example 19, and also horizontal force of 2 cwts. at the middle of the left-hand rafter. Assume the reaction at B vertical.

21. Example 20, assuming the reaction at A vertical.

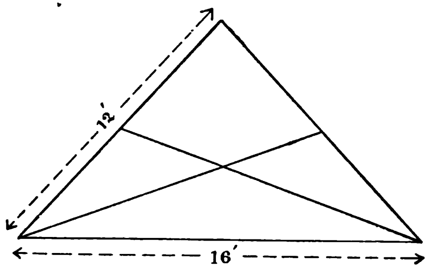
Remark.—In a large roof it is undesirable to fix *both* sides rigidly to the walls, for any change of dimensions, due to heat or cold, produces forces of great intensity (witness the bursting of water pipes in a frost) which might fracture the supports, or part of the roof. It is customary therefore to fasten one side firmly down and let the other side rest on rollers as shown in the figure, page 272.

22. Suppose the slope of the long rafters in Examples 18–21 to be $a°$. Determine, from the force diagram, formulæ for the stresses in the various bars.



23. Reactions vertical; span, 32 feet; length of rafters, 24 feet; length of short pieces, 4 feet; loads, 30 cwts. at apex and centres of long rafters.

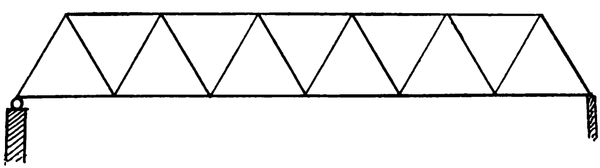
24. Same frame ; loads, 3, 4 and 5 cwts.



25. Reaction vertical ; long rafters, each 12 feet; cross rafters at middle of long rafters ; loads, 10 cwts. at apex and centres of long rafters.

In practice the 12-feet rafters would not be *jointed* at their middle points. It is customary however to treat them as jointed, the error being, speaking generally, on the side of safety. That is to say, we calculate the stress in the 12-feet rafters assuming them jointed. We choose beams capable of bearing the calculated stress if jointed. As they are in reality unjointed they are even stronger than need be.

The effect on the pins at the ends of replacing two jointed bars by a continuous bar is not so simple.



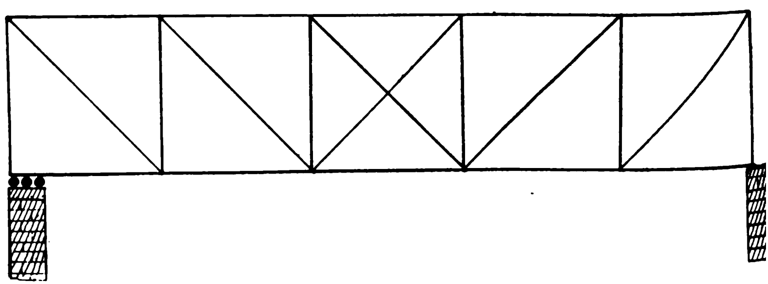
26. Reactions vertical. Load 1 ton at centre joint.

This structure is known as a Warren girder, from

Captain Warren, R.E., who first proposed its use for bridges. It had been used by Watt as a strong yet light *beam* for his steam engine.

In practice, the top and bottom horizontal bars, called flanges, are continuous, but the same assumption is made as in the preceding example. The triangles are usually equilateral and the reactions vertical.

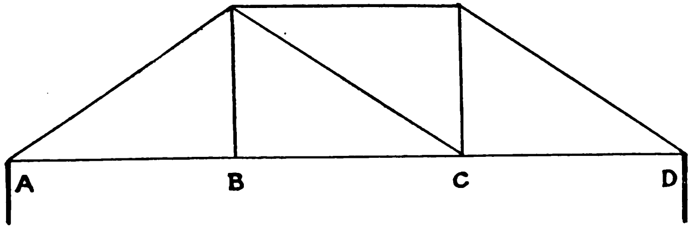
27. Draw the force diagram for the girder in the following case: (1) 2 tons at each joint of lower flange; (2) 2 tons at each joint of top flange; (3) 2 tons at each joint of top and bottom flanges; (4) a single load of 5 tons at the second joint of the bottom flange.



28. N-girder, reactions vertical. Slope of braces, 45°.

This is a modification of the Warren girder, designed to reduce the length of the struts in the cross-bracing to a minimum for a given depth of girder.

29. Draw the force diagram for the above girder for the following loads: (1) 5 tons at each joint of bottom flange; (2) loads in joints of bottom flange 10, 15, 5, 5—(*a*) from left to right, (*b*) from right to left. Reactions vertical.

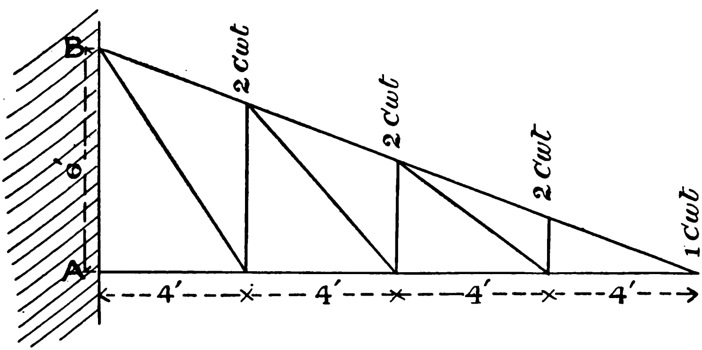


30. Vertical reactions at A and D; span 18 feet; height 8 feet; loads, 3 cwt. and 6 cwt. at the points of trisection B and C.

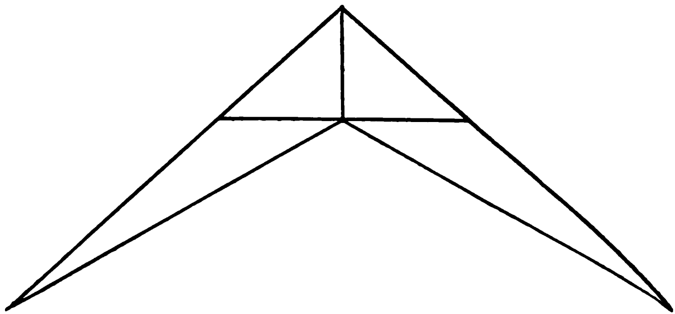
31. A rectangular frame 3 feet by 4 feet has two diagonals. If one diagonal is screwed up to a tension of 500 pounds, find the stresses in the remaining bars of the frame.

32. ABCD is a square frame: forces 1, 2, 3, P act outwardly at its angular points respectively; the direction of the force 1 bisects the angle at the point where it acts. Find by a construction the directions of the other three forces, and show that P does not differ much from $2\frac{1}{2}$.

33. The jib AB of a crane is 32 feet long, the post AC 15 feet, the tie BC 19 feet. The weight of 48 cwt. is attached to a chain which runs over a pulley at the end of the jib and then along the tie. Find graphically the stresses in the jib and the tie.

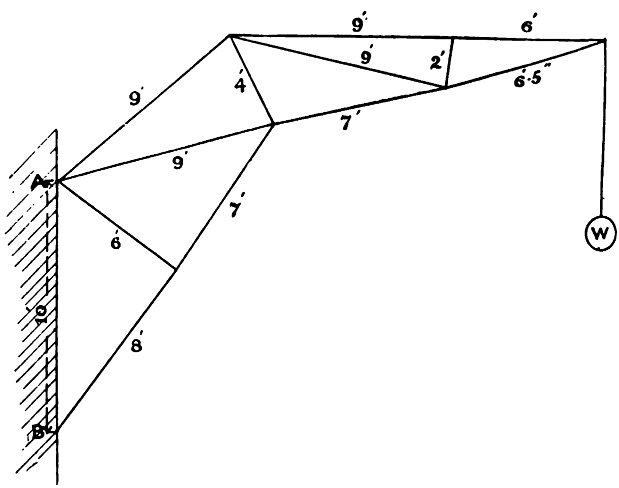


34. A cantilever, hinged to a wall at A and B, and its load are shown above. Draw the force diagram, and tabulate the stresses.



35. Vertical reactions, span 18 feet, height 9 feet, length of vertical bar 4 feet, loads 2 cwt. at apex and at middle of each long rafter.

36. Consider the same frame with loads of 2 cwt. at the apex and 3 and 5 cwts. at the centres of the long rafters.



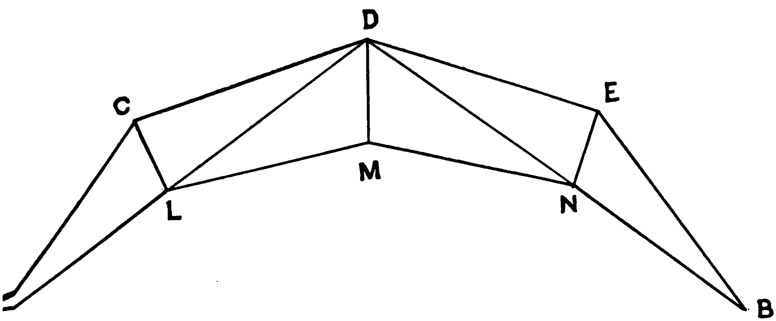
37. Frame hinged at A and B to a vertical wall. Examine the

effect of lengthening the 4-feet bar to 6 feet, altering the 7-feet bars so as to leave all the other joints in their former position.

38. A frame of five bars consists of two equal triangles, HKM, HKN, M and N being on opposite sides of HK, and in a horizontal line; HK = 10 feet, HM = HN = 40 feet, KM = KN = 35 feet; it stands on horizontal supports under M and N, and carries a weight of 2000 lbs. at K; draw a diagram for the stresses, and state the magnitude of each, and whether it is a tension or a thrust.

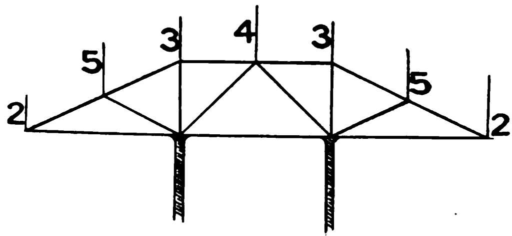
39. A small ring lies on a smooth horizontal square table, at a point which divides the line joining the middle points of two opposite sides in the ratio of 1 : 3. It is kept in this position by four strings, each of which is tied to the ring. The strings pass severally over pulleys at the four corners of the table and carry weights. The weights at each of the corners nearest to the ring are of 20 ozs. Find the weights at the other corners.

40. A spar inclined at 30° to the vertical carries at its upper end a small, freely-moving pulley wheel. A weight of 800 lbs. hangs by a rope passing over this wheel. The spar is supported by a tie rod inclined at 30° to the vertical. Find the tension set up in this tie rod, and the thrust produced in the spar when the pulling end of the rope is inclined at 45° to the horizontal.

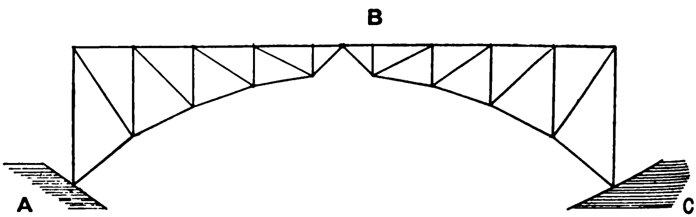


41. AB = 20 feet. Draw arcs of radii 10·5 and 13 feet passing through A and B. Place in each arc four equal chords, viz. AC, CD, DE, EB, and AL, LM, MN, NB respectively. The complete

frame thus determined is shown above. Suppose it loaded with 2 cwts. at C, D and E, and supported by vertical reactions at A and B. Draw the force diagram and tabulate the stresses in the bars.



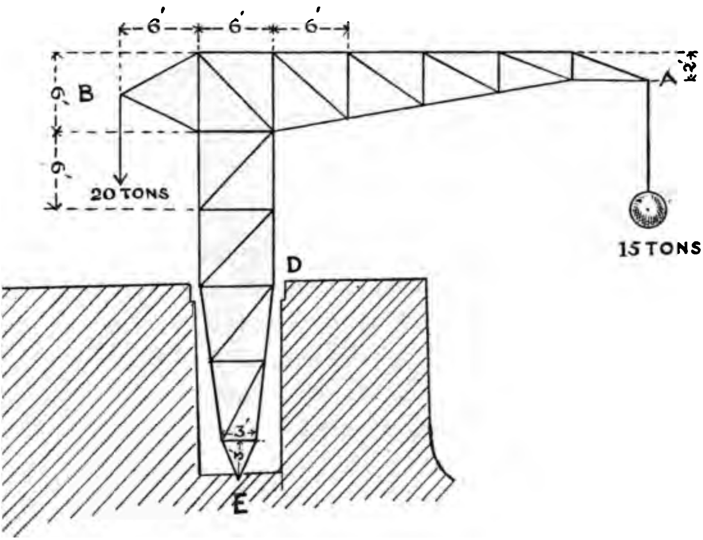
42. A roof-truss for a railway station platform is supported on two pillars and loads in cwts. are as shown. Find the stresses in the various bars of the frame.



43. An arch is freely hinged at A, B and C. A and C are fixed to masonry abutments. Ascertain the direction and magnitude of the reactions at A and C, and draw the force diagram : horizontal intervals, 12 feet ; length of verticals, 28, 18, 12, 8 and 6 feet ; loads, 1 ton at each top joint, including B. Assume the stresses in the inclined bars at B equal.

44. The framework of a crane is shown in the accompanying diagram. The load is at A ; a counterweight is at B ; E works in a fixed socket, and the framework is supported by a ring of rollers

which exert a horizontal thrust at D. Construct the force diagram and draw up a table of results.

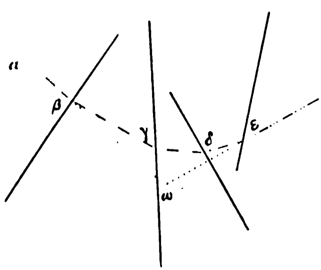


CHAPTER IX

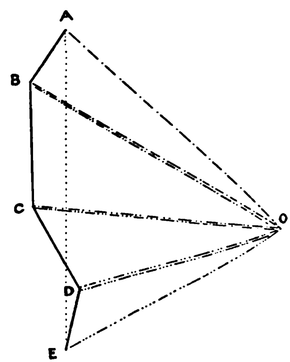
LINK POLYGONS

We shall now endeavour to determine by a graphical method the line of action of the resultant of several coplanar forces whose lines of action are not concurrent.

Suppose four forces P, Q, R, S to act along given lines, as in figure I. We take the case of four forces, but the method is perfectly general. The student should draw the following diagrams in coloured chalks, using one colour for P and its components and their lines of action, another for Q, and so on.



I.



II.

On any convenient scale draw lines AB, BC, CD, DE to represent P, Q, R and S.

Take *any* point O in the plane and join OA, OB, OC, OD, OE.

AO and OB	represent	forces	whose	resultant	is	P,
BO OC	„	„	„	„		Q,
CO OD	„	„	„	„		R,
DO OE	„	„	„	„		S.

Now take *any* point β in the line of action of P. Draw through β lines $\alpha\beta$, $\beta\gamma$ parallel to AO, OB—$\beta\gamma$ meeting the line of action of Q in γ.

Through	γ	draw	$\gamma\delta$	parallel to	OC,	meeting	line of	action of	R	in δ.
„	δ	„	$\delta\epsilon$	„	OD,	„	„	„	S	„ ϵ.
„	ϵ	„	$\epsilon\omega$	„	OE.					

P supposed to act at β may be replaced by its two components, represented by AO, OB and acting along $\alpha\beta$ and $\beta\gamma$.

Q supposed to act at γ may be replaced by its two components, represented by BO, OC acting along $\gamma\beta$ and $\gamma\delta$, and so on. The forces represented by BO and OB neutralise one another—so do those represented by CO and OC, and so on.

Thus the four forces P, Q, R, S may be replaced by—

(i) A force represented by AO acting along $\alpha\beta$,
(ii) „ „ „ OE „ „ $\epsilon\omega$.

Three cases may arise :—

I. If P, Q, R, S have a single resultant it is represented in magnitude and direction by AE and its line of action goes through the point of intersection of $\alpha\beta$ and $\epsilon\omega$.

II. If P, Q, R and S are in equilibrium the forc

polygon will close, *i.e.* the point E will coincide with the point A; and the lines $\alpha\beta$ and $\epsilon\omega$ will coincide.

III. If P, Q, R and S are equivalent to a couple—the point E will coincide with the point A; and the lines $\alpha\beta$ and $\epsilon\omega$ will be parallel, but not coincident.

The figure ABCDE is technically called the force polygon of the system. The figure formed by the lines $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\epsilon$, $\epsilon\omega$ is termed the **link polygon**, or, by some writers, the funicular polygon of the system.

The foregoing three cases may now be stated concisely as follows—

	Force Polygon.	Link Polygon.
I. For a single resultant .	Open	Open or closed[1]
II. For equilibrium . .	Closed	Closed
III. For a couple . .	Closed	Open

Where no confusion is likely to arise the figure OABCDE is sometimes referred to as the "force polygon."

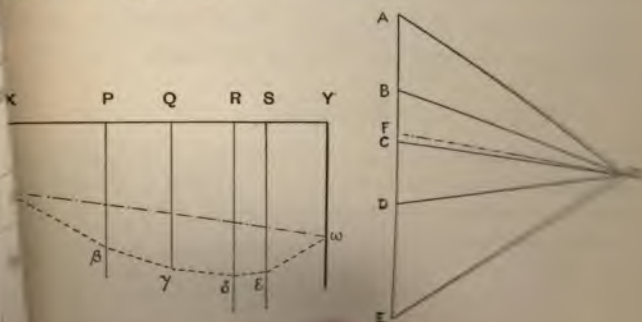
As the point O may be taken *anywhere* in the plane and the point β *anywhere* on the line of action of P it is clear that the construction may be carried out in very many ways.

With a little practice positions for O and β can be so selected and the *order* of the forces so chosen as to avoid the inconveniences caused by the link polygon not

[1] In this case the link polygon will be closed if O has been taken *anywhere on* the line AE.

falling within the limits of the paper, as for instance by the line $\beta\gamma$ being parallel, or nearly parallel, to the line of action of Q.

Reactions by a Link Polygon.—To find the vertical reactions X, Y, acting at two given points, which will equilibrate a given system of vertical loads P, Q, R, S applied at given points.



The system of forces [illegible], X, Y being in equilibrium any link poly[illegible] force polygon of the [illegible] must [illegible] closed [illegible]

Dr[illegible] rce [illegible] DE for P, Q, R, S [illegible]

[illegible] ction $\alpha\beta\gamma\delta\epsilon\omega$ of the [illegible]

[illegible] X in a,

[illegible] Y " ω.

[illegible] ygon for the whole [illegible] components of X must [illegible] of Y in $\omega\alpha$ and $\omega\epsilon$.

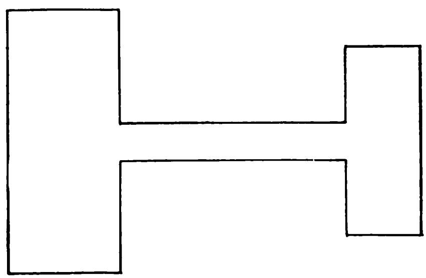
(4) Draw OF parallel to $a\omega$ to meet AE in F. The components of X must be FO and OA. Hence FA must represent X, and EF must represent Y.

Corollary.—(i) Observe that if $\alpha\beta\gamma\delta\epsilon\omega$ represents a frame, the forces P, Q, R, S, X, Y applied at the joints will keep it in equilibrium; and the figure OABCDEF is a Bow's diagram for the stresses in the framework. The force and link polygons may accordingly be lettered by Bow's method.

(ii) The point of intersection of $\alpha\beta$ and $\omega\epsilon$ is in the vertical line through the C.G. of the loads.

Examples.

1. A horizontal beam 10 feet long is supported at its ends A, B, and carries loads of 1, 3, 3, and 5 cwts., applied at points 2, 3, 5 and 7 feet from A. Determine the vertical reactions at A and B by a link polygon construction.



2. Figure 3 shows the shape of a piece of thin iron plate of uniform thickness to a scale of $\frac{1}{18}$. Determine the position of the C.G. of the plate, by constructing, by means of a link polygon, the line of action of the resultant of the weights of the individual rectangles.

Experiment 1.

Consider three forces of 1·5, 2, and 3·5 lbs. acting consecutively along the sides and hypothenuse of an isosceles right-angled triangle, side 4 inches. Determine the line of action of the equilibrant of these

forces, by a link polygon. Construct a model of the link polygon, and test whether the four forces applied[1] to the joints maintain it in equilibrium. Use thread for those sides of the link polygon which are in tension, and a light wooden rod for the sides which are in compression. Notice that you can ascertain from the diagram which pieces must be made of wood.

If several students try this experiment they will obtain a series of different link polygons, each of which will be in equilibrium under the given set of forces.

EXAMPLES.

3. Make use of the link polygon construction to find the resultant of forces of 4, 5, 4, and 5 pounds acting in order round the sides of a square of 2-inches side.

4. A, B, C are three points in a body all lying in one straight line. AB is 2 feet and BC 3 feet in length. The body is supported with ABC horizontal by three cords AP, BQ, CR, all lying in a vertical plane and attached to A, B and C respectively. The tension in AP is 60 pounds, in CR 30 pounds, the angle CAP is 150°, CBQ is 45°, ACR is 120°. Find the weight of the body by a force polygon, and the vertical line through the C.G. by a link polygon.

5. Show that a given force acting in a given line is the resultant of three definite components acting in given non-concurrent non-parallel lines.

[Hint: Draw any polygon with its vertices on the four given lines. This being the link polygon, draw a line to represent the given force, and draw the parallels which fix the position of O the pole.]

6. A horizontal beam is supported at one end A and anchored at the other end B, and acted on by given forces. Show how to determine the reactions.

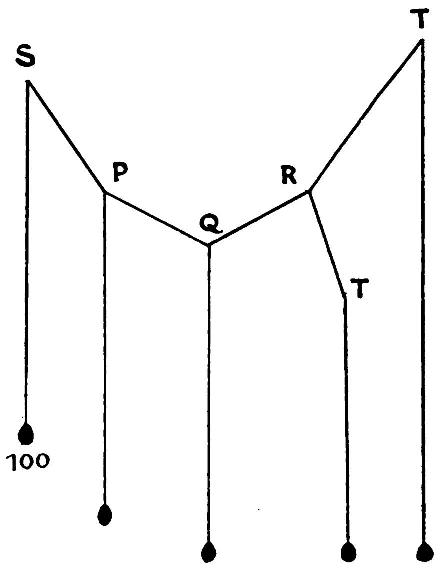
[Hint: Draw the force polygon for the given forces. Take any pole O and join up as usual. Start the link polygon from the point B.]

---

[1] The link polygon may be horizontal, or vertical, and the forces be applied by spring balances, or preferably by weights hanging over pulleys fixed to a framework.

7. Draw a square ABCD, with AB horizontal. A weight o pounds is to be supported by a force at A, inclined at 45° to a force at C passing also through the middle point of AD, a foı B making 60° with BA. Determine—by drawing a link polygoı then a force polygon—the magnitude of these three forces.

8. The figure represents a number of strings knotted togeth P, Q and R, and passing over smooth pulleys at S, T and T. Iı vertical strings carry weights, and that hanging from the pı S is 100 grams, find the other weights. [Aı



Suspension Bridge.—Suppose a uniform platform hı by equidistant vertical rods, from a chain. If we negl the weight of the rods and the weight of the chain, comparison with the weight of the platform, we can reaı determine the form of the chain.

The chain will be supposed to consist of a series

straight bars, each bar going from the top of one tie rod to the top of the next.

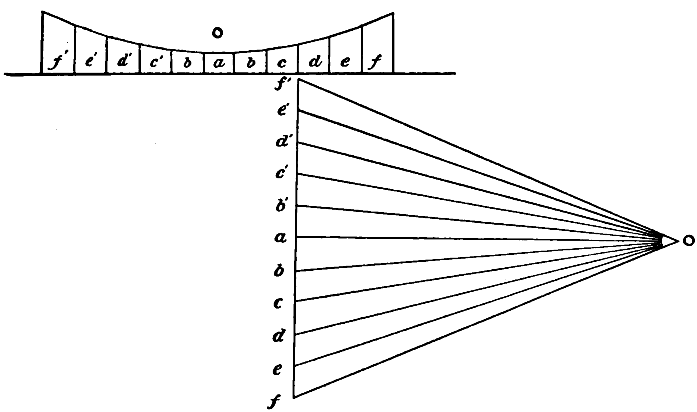
The conditions just mentioned are approximately realised in an ordinary suspension bridge.

Some numerical examples will render the method of solution clear.

EXAMPLE.

9. In a small suspension bridge of 66 feet span the platform is supported by ten vertical rods (exclusive of the end piers) on each side at equal intervals—the rods being attached to two chains, one on each side of the platform. Given that the tension in each vertical rod is 10 cwts., and that the tension in the central link of the chain is 6 tons, find the tension in the other links of the chain, and their inclination to the vertical.

Treating the chain as a frame and lettering the diagram by Bow's notation we can draw the triangle of forces for *aob′*.



Make *ab′* vertical to represent 0·5 ton, and *ao* horizontal to represent 6 tons.

Next, draw the triangle of forces for the joint $b'oc'$. Continuing we finally obtain the complete force polygon for the frame.

The directions of the links of the chain have now been found, for ob' in the frame diagram is parallel to ob' in the force diagram, and so on.

EXAMPLES.

10. Show that the tangents of the angles of the inclination of the links to the horizontal are in arithmetical progression, and write down their values.

$$[\tan\theta_1 = \frac{1}{12},$$
$$\tan\theta_2 = \frac{2}{12} \text{ and so on.}]$$

11. Find the length of each link by measurement from the diagram, and check by calculation.

[Here $ob' = 6 \sec\theta_1$,
$oc' = 6 \sec\theta_2$, etc.]

12. Find the tensions in ob', oc' . . . etc., by measurement from the force diagram, and check by calculation.

$$[T_{bo'} = \sqrt{6^2 + \frac{1}{2^2}},$$
$$T_{oc'} = \sqrt{6^2 + 1^2},$$
or
$$T_{ob'} = 6 \sec\theta_1,$$
$$T_{oc'} = 6 \sec\theta_2.]$$

13. Show that if the tension in the central link is any given fraction, say $\frac{2}{3}$ of the total load, the force polygon can be at once drawn.

14. Find the lengths of the several vertical rods (given that length of $ab' = 2$ feet) from the frame diagram, and check by calculation.

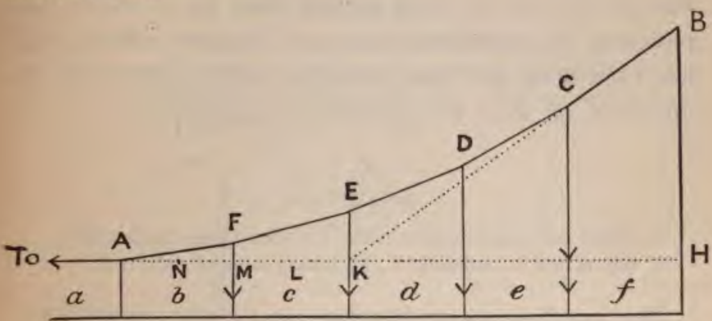
$$[b'c' = ab' + 6\tan\theta,$$
$$= 2 + \tfrac{1}{2} = 2\tfrac{1}{2}.]$$

In practice the problem presents itself in a somewhat different way.

EXAMPLE.

15. Given the height of the "piers" (uprights to which the ends of the chain are attached) to be 12 feet, the span 66 feet as before, the same number of vertical rods, of which the central ones are each 2 feet long, and the tension in each 0·5 ton as before. Calculate the tension in the central link of the chain, and find (1) the tensions in all the links, (2) the inclination of each link to the horizontal, (3) the length of each link, (4) the length of each vertical tie rod.

Our first step must be to calculate the tension in the central link.



Consider the equilibrium of the portion of the chain comprising half the top link, half the central link, and the intermediate links.

The forces acting on this portion of the chain are (1) the pull, say T_0 of the central link, which by symmetry must be horizontal, (2) the vertical pulls of the five tie rods, each $\frac{1}{2}$ ton, (3) the pull T in the topmost link—the line of action of T passes through B, but its direction is unknown.

The resultant pull of the five tie rods, *i.e.* 2·5 tons, acts in the rod *cd*. The lines of action of T_0 and the *weight are known,* and these meet in a known point,

say K. Hence the line of action of T must also pass through K, and consequently the first link must lie along the line BK, and is therefore represented by BC in the frame diagram.

The triangle of forces can now be drawn, and T_0 and T determined. Note that BKH is a ready drawn triangle of forces. This is important in military bridging. The actual pattern of the chain, roadway and tie rods will be laid out full size on level ground close by, to enable the pieces to be constructed and put together. When this has been done the force triangles such as BKH can be measured off from the pattern on the ground.

EXAMPLES.

16. What is the inclination of the end link to the horizontal?
17. Show how to calculate T_0 by taking moments about B.

The complete force diagram can now be drawn, after Bow's method, as in the foregoing example. The positions of the remaining links may be determined from the Bow's diagram; or by applying the same reasoning as before to the forces acting on the portion of chain comprising half of the middle link, half of the link CD, and the intermediate links.

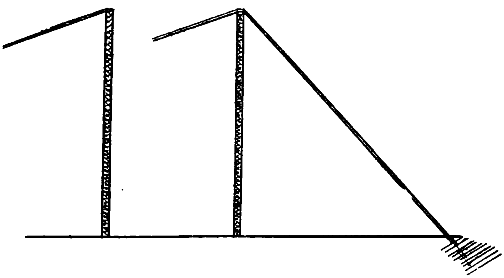
18. Examine the effect of decreasing the "dip" or depth of the central link below the tops of the piers to (*a*) 8 feet, (*b*) 6 feet.
19. Express the ratio of the greatest to the least tension in the chain, in terms of the ratio dip : span.

If there is an *odd* number of vertical rods there is no horizontal link. This case will be most easily dealt with

;ining a very short horizontal link put in at the
of the chain, and the central tie rod divided into
:ions, one at each end of this horizontal link. The
in each of these two portions will be half the
in any other vertical tie rod. The value T_0 can
found, as before, and the force polygon drawn.
e link polygon, *i.e.* the form of the chain, can be

EXAMPLES.

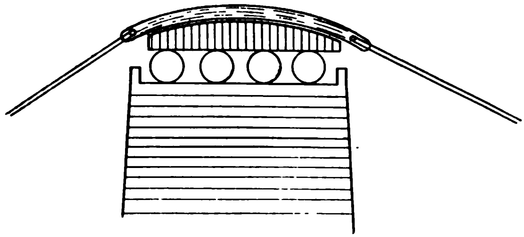
ɔpose seven equidistant vertical tie rods, the tension of ts., to support a platform 24 feet long, the middle of the ıg 4 feet below the tops of the piers. Determine the form .in and the tension in each link.



:.—In practice the chain is not fastened to the top ɪier, but is led over it and anchored down to the as in the right-hand figure.

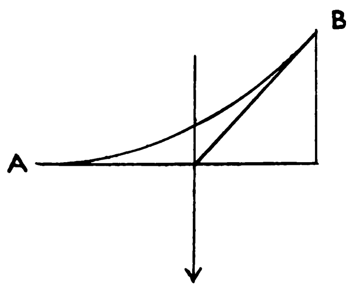
he tension in the topmost link is 12 tons, and the height r 18 feet, what is the moment tending to upset the pier, the chain fastened to the top of it and inclined at 75° to it? in the case of the last example, the chain is led over the pier, at what point of the ground should it be fastened in t there may be *no* moment tending to upset the pier? ıe tension to be the same on either side of the pier.
ain usually rests on a roller or "saddle" on the top of

the pier to give the small amount of play necessary to allow for expansion.



23. Suppose the topmost link in Ex. 21 fastened to the saddle, and another chain fastened on the other side of the saddle, and leaving the pier at an inclination of 45° The saddle rests on rollers, which are on the top of the pier. What must be the tension in this chain if the saddle is in equilibrium? Will there be any upsetting moment on the pier in this case?

The assumption that the load on the platform is uniformly distributed is very closely realised as far as the weight of the roadway or "dead" load on an ordinary suspension bridge is concerned. But a moving or "live" load on the bridge interferes with the truth of this assumption, and hence a suspension bridge is not suited for the passage of heavy live loads, heavy, that is to say, in comparison with the dead load. Thus the suspension type is unsuitable for railway traffic.



If a wire is stretched nearly straight the weight of any portion, AB, of it is very nearly proportional to the horizontal distance

between A and B, and so the vertical through the C.G. of the portion AB may be assumed to fall midway between A and B, and will be concurrent with the tangents to the wire at A and B.

EXAMPLES.

24. Suppose the average slope of the wire to be 5°, and the weight 1 lb. per foot, compare the real weight of a 100-yards span with the weight given by the above assumption.

25. A wire weighing 7 pounds stretched between two points 200 feet apart in the same horizontal line sags 10 feet in the centre. Consider the forces acting on half the wire, and show that the tension at the lowest point is 17·5 lbs., and find the tension at the points of attachment.

26. If the wire in the preceding question would break under a tension of $1\frac{1}{2}$ tons, what is the least possible dip at the centre?

27. In experimenting on a model it is found that a wire of 0·1 inch radius can just be stretched without giving way between two points 100 feet apart, with a dip of 1 foot at the centre. Find whether a wire of the same material 1 inch diameter in section can be stretched across a span of 1,000 feet with a dip of 10 feet at the centre.

28. A span of telegraph wire weighs 100 lbs., and the two ends are at the same level and inclined at 20° to the horizontal. Consider the forces acting on the part of the wire between the middle point and one end, and thus find the pull in the wire at the middle and at the end.

29. If steel wire, weighing 0·29 lbs. per cubic inch, will bear a tension of 50 tons per square inch before it breaks, find the greatest possible distance over which a steel wire of *any* thickness could be hung with (*a*) dip of $\frac{1}{10}$ at centre, (*b*) dip of $\frac{1}{20}$ at centre, without breaking under its own weight.

The fact that increasing the diameter of the wire does not increase the possible span should be clearly realised. The reason is that the weight increases in exactly the same proportion as the strength.

30. A telegraph wire, weighing $\frac{3}{4}$ oz. per foot, is fastened to the top of two poles of equal height, 80 yards apart on the same level. The dip at the centre is 2 feet. What is the force (1) tending to

turn over the pole, (2) increasing the pressure of each on the ground?

31. A suspension bridge, 160 feet long, and weighing 2 tons per foot, is carried by two cables. These cables make angles of 63° with the piers, and droop at the middle 20 feet below the heads of the piers. Each pier is supported by two back stays inclined at 40° to the horizontal. Draw diagrams of the forces acting (1) on one-half of the bridge, and (2) at the head of a pier, and find the pull in the chain at the middle, and at the pier, and in the back stay.

32. Solve graphically Exs. 80 and 81, p. 240.

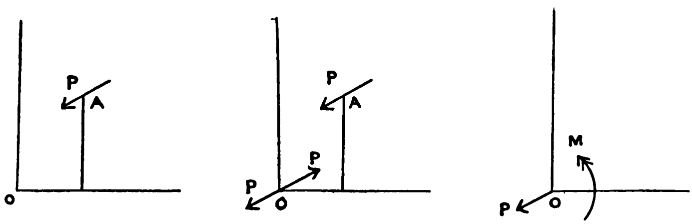
CHAPTER X

GENERAL CONDITIONS OF EQUILIBRIUM

IN the chapter on the link polygon we have shown that a system of forces in one plane acting on a rigid body may either (1) be reduced to a single force—their resultant—or (2) be reduced to a couple, or (3) may be in equilibrium.

We now deal with the same problem by methods of calculation. In actual practical examples, just as in the case of three forces, sometimes calculation and sometimes construction will be the more suitable method.

Proposition 1.—A single force acting at any point of a rigid body may be replaced by an equal force acting at any chosen point together with a couple.



I II III

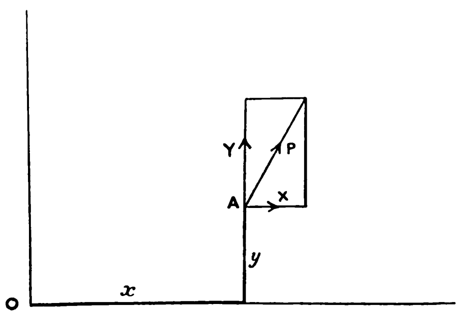
Let A be the point at which the given force P acts, and

let O be the chosen point. Apply at O a force equal to P, and a force equal and opposite to P. This of course does not affect the equilibrium of the rigid body. The given force P, and the force equal and opposite to it at O form a couple, and thus P at A may be replaced by an equal force P at O and a couple.

The most convenient way of finding the magnitude of this couple is, generally speaking, to replace P by two components X and Y parallel to two rectangular axes through O.

We then have—

Moment of couple about any point in its plane
= Moment of couple about O,
= Moment of P about O,
= Sum of moments of X and Y about O,
= $Yx - Xy$.



If x and y denote the distances of A from O measured parallel to the axes x and y are called the "co-ordinates" of A.

EXAMPLES.

1. A person is endeavouring to turn a horizontal handle CBD fixed at right angles to a vertical rod AB, by applying a horizontal force through C at right angles to CBD. Show that in addition to a couple twisting the rod AB there is a lateral thrust tending to bend the rod.

2. In the previous example let B be the middle point of CD, and suppose equal and opposite horizontal forces applied at C and D. Show that there is in this case no lateral thrust tending to bend the rod.

[Examples 1 and 2 are of importance in connection with such practical problems as cutting a screw on a long thin rod.]

3. A brass rod 18 inches long is fixed vertically upright, with its lower end in a vice. A screw is to be cut on it, the screw-cutting tool being worked by a horizontal handle 12 inches long, to one end of which a horizontal force of 20 lbs. is applied. Find the moment of the force tending to snap the rod off short at the vice.

Reduction of a System of Forces.—*Proposition* 2.—Any number of forces in one plane acting on a rigid body may be reduced to a single force acting at any point we please in the plane, together with a certain couple.

Let P_1, P_2, P_3 . . . be the forces, and suppose that they are applied at points whose co-ordinates are x_1y_1, x_2y_2, x_3y_3, etc.

(1) Replace P_1 by its components X_1Y_1, parallel to the axis, and similarly P_2 . . . by its components X_2Y_2 . . . and so for the remaining forces.

(2) By proposition 1 replace X_1Y_1 at (x_1y_1) by X_1Y_1 at O, and a couple whose moment is $Y_1x_1 - X_1y_1$

(3) Repeating this step for each pair of components in turn, the original set of forces is reduced to—

The set of forces X_1, X_2, X_3 . . . acting at O along Ox,
" " Y_1, Y_2, Y_3 . . . acting at O along Oy,

and the set of couples—

$$Y_1x_1 - X_1y_1,\ Y_2x_2 - X_2y_2, \text{ etc.}$$

(4) **Reduction of a System of Forces.**—We have already used the symbol Σ (X) to denote $X_1 + X_2 + X_3$ . . . We may state our results thus—

The set of forces P_1, P_2, P_3 . . . may be reduced to

(i) A force $\mathbf{X} = \Sigma$ (X) along Ox,
(ii) A force $\mathbf{Y} = \Sigma$ (Y) along Oy, and
(iii) A couple $\mathbf{M} = \Sigma (M) = \Sigma (Yx - Xy)$.

(5) The forces $\mathbf{X}$ and $\mathbf{Y}$ may be replaced by their resultant, a force $\mathbf{R}$ acting at O, where $\mathbf{R}^2 = \mathbf{X}^2 + \mathbf{Y}^2$. From this proposition several important deductions follow immediately.

I. Any system of coplanar forces acting on a body may be balanced by a force acting at any chosen point of the plane, together with a certain couple.

EXAMPLE.

4. Can the single force have any direction we please?

II. Since a single force $\mathbf{R}$ can never balance a couple $\mathbf{M}$, it is clear that if the original set of forces is in equilibrium, we must have

$$\mathbf{R} = 0,\ i.e.\ \mathbf{X}^2 + \mathbf{Y}^2 = 0, \text{ and } \mathbf{M} = 0.$$

that is, in other form, since if $\mathbf{R}$ vanishes both $\mathbf{X}$ and $\mathbf{Y}$ must vanish—

$$\mathbf{X} = \Sigma\,(\mathrm{X}) = 0,$$
$$\mathbf{Y} = \Sigma\,(\mathrm{Y}) = 0,$$

and
$$\mathbf{M} = \Sigma\,(\mathrm{Y}x - \mathrm{X}y) = 0.$$

III. Conversely, if a system of coplanar forces acting on a rigid body satisfy the equations—

$$\Sigma\,(\mathrm{X}) = 0,$$
$$\Sigma\,(\mathrm{Y}) = 0,$$
$$\Sigma\,(\mathrm{Y}x - \mathrm{X}y) = 0,$$

they must be in equilibrium.

This most important result simply amounts to the following common-sense statement: "If a set of forces keep a body in equilibrium they can have no tendency to move the body in *any* direction, and hence the sum of their resolved parts in any chosen direction must vanish."

"They can also have no tendency to turn the body about *any* point whatever, and so the sum of their moments about any chosen point must vanish."

EXAMPLE.

5. Show that if the sum of the resolved parts of the forces in any two directions vanishes, then the sum of the resolved parts in any direction whatever must vanish.

We see, therefore, that in order to be certain that the sum of the resolved parts in *any direction whatever* must vanish, it is only necessary to verify that the sum of the resolved parts in each of *two chosen* directions does vanish.

The classical student will be reminded in connection with the words in italics of the difference between "quivis" and "quidam."

IV. Let us next consider what inference can be drawn from the fact—

"The sum of the moments about a certain point O of a set of forces vanishes."

We at once infer that the set of forces is not reducible to a couple: for there is no point whatever in the plane about which the moment of a couple vanishes. We infer that the forces *may* be in equilibrium; or may have a resultant whose line of action goes through O.

Next, let the further information be given that the sum of the moments of the forces about two points O and O′ each separately vanish. If the forces are not in equilibrium, the line of action of their resultant must pass through O and also through O′.

If, then, finally we are given that the sum of the moments of the forces about three points O, O′ and O″, *which are not collinear*, each separately vanish, the forces must be in equilibrium.

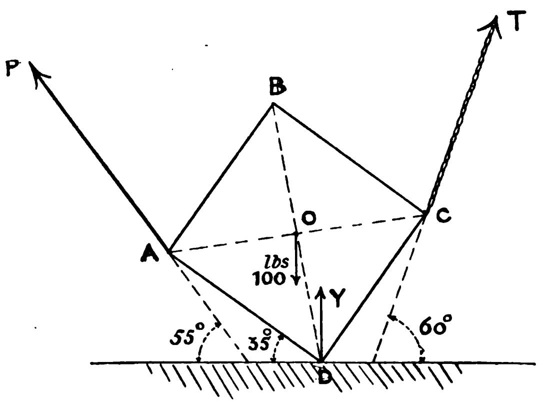
Examples.

6. Show that if the algebraic sum of the moments of the system of forces about each of three non-collinear points, A, B and C, in a plane vanishes, then the algebraic sum of the moments about any point whatever in the plane will vanish.

7. State the conditions that must be satisfied by a number of forces in one plane which are in equilibrium. Would it be sufficient if the algebraic sum of the moments of all the forces about two points in the plane were zero?

As a general rule, we have not to deal with a set of known forces, and to find whether they are in equilibrium; but, knowing that a body is in equilibrium under the action of certain forces, we desire to find what relations must hold between the forces. An example will illustrate this.

The figure below shows a packing-case ABCD 2 feet cube, weighing 100 lbs., being lowered by a chain on to the floor. A man, by means of a rope attached at A, lifts the case so that the edge at D reaches the floor first.



Taking the dimensions and angles from the figure find the tension (T) in the chain, the tension (P) in the rope and the pressure (Y), which is assumed to be vertical of the case on the floor.

The body in question and the forces acting on it are indicated in the figure. The next step is to write down the three statical equations.

Resolve horizontally—

$$T\cos 60^\circ - P\cos 55^\circ = 0$$

Resolve vertically—

$$T\sin 60^\circ + Y + P\sin 55^\circ - 100 = 0$$

Take moments about the centre of the case O—

$$T\,.\,OC\sin 50^\circ + Y\,.\,OD\sin 10^\circ - P\,.\,OA\sin 65^\circ = 0$$

No statical equations independent of these can be given.

Collecting these equations—

$$T\cos 60^\circ - P\cos 55^\circ = 0 \quad . \quad . \quad . \quad (1)$$
$$T\sin 60^\circ + P\sin 55^\circ + Y = 100 \quad . \quad . \quad . \quad (2)$$
$$T\sin 50^\circ - P\sin 65^\circ + Y\sin 10^\circ = 0 \quad . \quad . \quad . \quad (3)$$

we notice that we have three equations to find three unknown quantities.

It is important to notice that when the statical equations have once been written down the question becomes a question of algebra—that of finding some unknown quantities from a sufficient number of equations.

Substituting the numerical values of the co-efficients, the equations become—

$$0{\cdot}5\,T - 0{\cdot}574\,P = 0,$$
$$0{\cdot}866\,T + 0{\cdot}818\,P + Y = 100,$$
$$0{\cdot}766\,T - 0{\cdot}906\,P + 0{\cdot}17\,Y = 0,$$

and solving these equations we find—

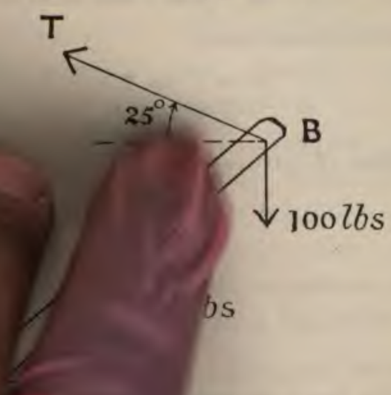
$$T = 58{\cdot}2 \text{ lbs.}, P = 50{\cdot}8 \text{ lbs.}, Y = 8{\cdot}1 \text{ lbs.}$$

EXAMPLES.

8. AB is a lid of a box 3 feet long, weighing 15 lbs., which is kept open at an inclination of 45° to the horizon by a *smooth* horizontal rail C, which is $2\frac{1}{2}$ feet from the hinge A. The end B is pulled horizontally by a force of 20 lbs. Find the pressure on the rail, and the direction and magnitude of the force on the hinge.

9. A uniform rod AB 4 ft. long, weight 50 lbs., can turn about a horizontal axis through its lower end A. A rope attached to B is inclined at 25° to the horizontal. A weight of 100 lbs. is suspended from B. If in the position of equilibrium the rod is inclined at 40° to the horizontal, find the tension of the rope and the reaction of the axis.

If we replace the loads of 50 lbs. and 100 lbs. by their resultant, the forces acting on the rod reduce to three in number, and a graphical solution may be readily obtained. If, however, we desire to calculate the results, it is better to suppose the reaction at A replaced by its horizontal component X and its vertical component Y.



librium is shown in the

Resolve horizontally—

$$X - T \cos 25^\circ = 0 \quad (1)$$

Resolve vertically—

$$Y + T \sin 25^\circ - 150 = 0 \quad (2)$$

Take moments round A—

$$4\,T \sin 65^\circ - 50 \times 2 \cos 40^\circ - 100 \times 4 \cos 40^\circ = 0 \quad (3)$$

Example.

10. What advantage is there in choosing A as the point about which to take moments, rather than the point B, or the centre C of the rod?

Equation (3) gives—

$$4\,T \sin 65^\circ = 500 \cos 40^\circ,$$

or

$$T = \frac{125 \cos 40^\circ}{\sin 65^\circ},$$
$$= 105{\cdot}7 \text{ lbs.} = 106 \text{ lbs. nearly.}$$

Equation (1) now gives—

$$X = T \cos 25^\circ,$$
$$= 95{\cdot}6 \text{ lbs.} = 96 \text{ lbs. nearly,}$$

and equation (2) gives—

$$Y = 150 - T \sin 25^\circ,$$
$$= 150 - 44{\cdot}5 = 105 \text{ lbs. nearly.}$$

Thus the reaction at A is the resultant of a horizontal force of 96 lbs. and a vertical force of 105 lbs.

11. Find this resultant in magnitude and direction.

[Notice that if only the tension of the string were required a single statical equation would give it.]

The question may be asked, what are the advantages

and disadvantages of the graphical method as compared with the method of calculation?

The choice of method is a matter of judgment in any particular case, unless, indeed, an examiner has expressed a preference.

We may mention a few of the merits and defects of the graphical method.

Pro.

(i) Any data are, generally speaking, easy to deal with: so that troublesome calculations of the lengths of lines and the magnitudes of angles are avoided.

(ii) The accuracy attainable is, in most cases and with careful drawing, quite as great as the accuracy with which the data are known.

(iii) In any practical problem it will nearly always be necessary to make a drawing to scale of the objects dealt with, as a guide to the constructor, and when this has been done it is usually easy to draw the force diagram.

Con.

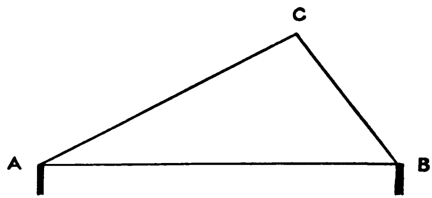
(i) In certain cases it is not easy to get an accurate result graphically, as for instance when very small angles have to be dealt with, or when some of the magnitudes involved are large and others small.

(ii) Any variation in the data generally renders it necessary to work out a fresh graphical solution, whereas calculation will always give a general formula, which may be easily worked out numerically.

Errors in Data.—It might be thought that the method of calculation was greatly superior in accuracy. If the original data were known with very great accuracy, which practically never happens, this view would be correct.

The student might think that if his data are uncertain to the extent of 5 per cent., and his method to the extent of 1 per cent., that his final results would be 6 per cent. out. They might *possibly* be so; but the probability is that the error in the final results would not exceed $\sqrt{5^2 + 1^2}$ or $\sqrt{26}$ per cent., that is say 5·1 per cent.

The matters which require attention in dealing with the equilibrium of a body which is made up of two separate parts will be best indicated by an example.

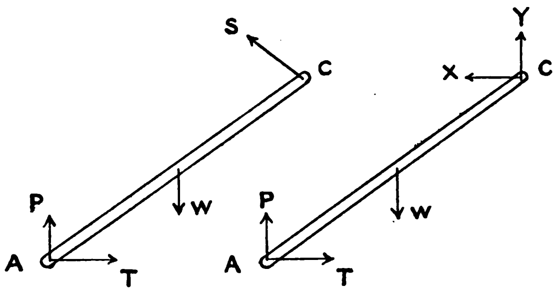


EXAMPLE.

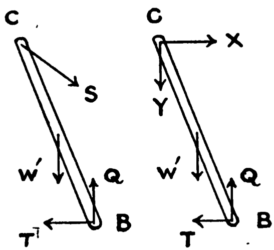
12. Consider the "roof" known as the Common Roof with unequal rafters. Let each rafter be uniformly loaded, supported by the vertical reactions of the two walls and held in position by a light tie rod AB (whose weight is put out of consideration) which exerts horizontal pulls on the rafters at A and B. Given the shape of the roof and the loads on the rafters, find the pressures on the walls, the tension of the tie rod, and the reaction between the rafters at C.

The two rafters AC and BC form a body which is in

equilibrium under certain forces. The rafter AC by itself is in equilibrium and so is the rafter BC.



I.



II.

The forces acting on AC are (Fig. I.)—

(1) the load W vertically,

(2) the vertical reaction P of the wall,

(3) the horizontal pull T of the tie-rod,

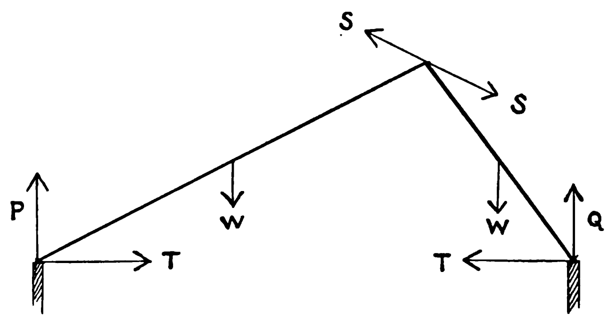
(4) the "action" S of the rafter BC on the rafter AC.

As S is unknown in direction, we will replace it by its horizontal and vertical components as in the right-hand figure.

The forces acting on BC are in like manner indicated in Fig. II.

The "reaction" of AC on BC is equal to the "action" S of BC on AC, and acts in the same line, but in the opposite direction. Consequently S may be replaced by components X and Y of the same magnitude as in Fig. I, but acting in opposite directions.

The forces acting on the roof as a whole consist of all the preceding forces. But the action S and the reaction S being equal and opposite and acting at the same point are in equilibrium by themselves, and will disappear from any equation of equilibrium for the roof as a whole which we choose to write down.



In like manner T and T disappear, and therefore the forces shown in Fig. III opposite are in equilibrium.

We will suppose the shape of the roof specified by the height h of the apex above the top of the walls and the angles of inclination α and β of the two rafters.

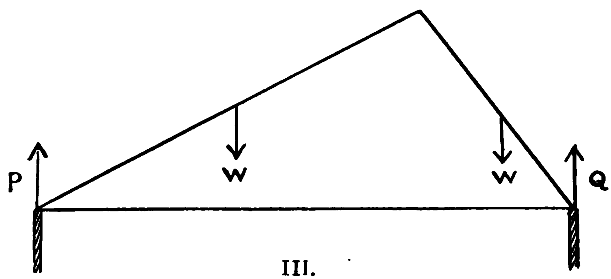
We can write down—

3 equations for the equilibrium of AC . . Fig. I.
3 „ „ „ BC . . Fig. II.
2 (for all the forces happen to be vertical) for the equilibrium of AC and BC . . . . . Fig. III.

Example.

13. Write down the eight equations. How many of them are independent?

It is not however necessary to write down all the equations in order to find the unknown forces.



III.

Fig. I. Take moments about C—

$$Ph \cot a - W.\frac{h}{2}\cot a - Th = O \quad . \quad . \quad . \quad (1)$$

Fig. II. Take moments about C—

$$Qh \cot \beta - W'.\frac{h}{2}\cot \beta - Th = O \quad . \quad . \quad . \quad (2)$$

Fig. III. Resolve vertically—

$$P + Q = W + W \quad (3)$$

Rewriting these equations after multiplying 1 and 2 by tan α and tan β respectively, we have—

$$P = \frac{W}{2} + T \tan \alpha,$$

$$Q = \frac{W'}{2} + T \tan \beta,$$

$$P + Q = W + W';$$

whence, immediately—

$$T = \frac{W + W'}{2} \cdot \frac{1}{\tan \alpha + \tan \beta}.$$

EXAMPLES.

14. Find the values of P, Q, X and Y.

[No universal rule can be given for choosing the most suitable equations. A good general rule is to endeavour to pick out equations, each of which will contain only one unknown quantity.]

15. If AC = 15 feet, BC = 8 feet, AC = 17 feet, find h, α and β by drawing and by calculation.

16. If in Example 15 W = 15 cwts., W′ = 8 cwts., work out the value of T.

Summary.—We will collect together the principal suggestions for the solution of problems which have been put forward in the preceding sections.

(1) Specify carefully the body, or the portion of the body, whose equilibrium you are considering.

(2) Specify the forces which act on the body, drawing a diagram showing the direction of each. If the body consists of two or more distinct portions or pieces, draw a separate diagram for each piece. It will generally be desirable to replace a force whose direction is unknown

by two components X and Y in known directions at right angles.

The sign of the numerical values found for X and Y will indicate whether they act in the direction assumed, say to the right, or in the opposite direction, for -7 to the right means of course 7 acting to the left. Hence it is not necessary to spend much time at the outset in considering which way a force acts, although a little thought will often settle the point; provided that if an "action" is assumed to act one way, the corresponding "reaction" must be taken to act the opposite way.

(3) Remember that, for each body or separate portion of a body that we consider, the relations between the forces which keep it in equilibrium may be completely expressed by three statical equations.

Hence though it is right to consider which statical equations can be most usefully employed it is perfectly useless to write down more than three for each body.

(4) Try to choose statical equations which each involve only one unknown quantity.

(5) *Internal Reactions.*—Remember when dealing with a body which consists of two pieces, A and B, that the action of A on B is equal and opposite to the action of B on A; and is an "external" force as regards B, but is balanced by the reaction of B on A if we are considering B and A together as one body.

Failure to realise this is the most frequent source of error in working out statical problems. The mistake is not a mistake in mathematics, but a mistake of fact.

If we are considering the equilibrium of A, then the

action of B on A is one of the forces to be reckoned with.

If we are considering the equilibrium of B, it is the action of A on B that is to be reckoned with—a mistake on this point is like reckoning my butcher's bill as part of *my* property.

Again if we are considering the equilibrium of A and B together as a whole, the action of A on B and the action of B on A may *both* be counted if we like, though as they are equal and opposite they balance one another, and may just as well be omitted.

Illustration.—A and B are partners and A has lent B £100. If A wishes to include the £100 as an asset, B must include it as a debt in reckoning up their joint possessions, and it will be simpler to leave it out altogether. A is right in treating the £100 as an asset if he is reckoning up his own private balance-sheet.

EXAMPLES.

17. A ladder 60 feet long, weighing 200 pounds, whose centre of gravity is 25 feet from its lower end, stands on a smooth floor, and leans against a smooth vertical wall. The ladder is inclined at 70° to the horizontal and is kept from slipping by a rope tied to a point 5 feet from the end of the ladder and fastened to a point in the wall. The inclination of the rope to the horizontal is 15°. Find the tension of the rope.

18. A pair of steps consists of two equal ladders each 5 feet long, and weighing 15 pounds, hinged together at the top. The steps stand on a smooth horizontal floor and are kept from slipping by a horizontal cord 2 feet long, connecting points 1 foot from the lower end of each ladder. Find the tension of the cord, when a man weighing 200 pounds stands at the middle of one ladder.

19. Three equal uniform rods AB, BC and CD, each of length 1 foot, are freely hinged together at B and C. Find how far apart two smooth pegs should be placed in the same horizontal line so that the rods may rest on them and remain in a straight line.

20. If the pegs in Example 19 are placed 14 inches apart determine the position of equilibrium and the reactions at B and C.

21. A ladder rests at an inclination θ against a retaining wall whose inclination is α. Supposing the ladder uniform and of weight W, and the wall and ground smooth—find the horizontal force which must be applied at the foot of the ladder to prevent slipping.

22. A rod AB, length 6 feet, weight 40 lbs., can turn about its lower end A, and is held up by a string CD, length 5 feet, fastened to a point C 5 feet above A, and to a point D in the rod. Calculate the tension of the string in terms of x, the distance AD, and tabulate the results, when $x = \frac{1}{2}$, 1, 2, 3, 4, and 5 feet.

23. A hollow cylinder radius 1 inch, weight 1 ounce, is open at both ends, and stands on a table. Two bagatelle balls, radius $\frac{3}{4}$ inch, are placed inside the cylinder. Find the weight of each ball if the cylinder will just not upset.

24. Two uniform rods, AB, BC, of length 16 inches and 12 inches respectively, are hinged together at B. The ends A and C are fixed to points 10 inches apart in a vertical line. Find the reactions at A, B and C, supposing the rods to weigh 16 lbs. and 12 lbs. respectively.

25. Three uniform rods AB, BC, CD, of length a, $2a$, a, respectively rest on a horizontal cylinder of radius r, AD and the middle point of BC being in contact with the cylinder and BC being horizontal. Find the reaction between BC and the cylinder.

26. Six drain pipes are stacked in a pile, three in the bottom row, two in the next row, and one on top. If each weighs 10 lbs. find the pressures between them, and the horizontal forces which must be applied to the outermost ones to prevent collapse.

27. A uniform rod AB, weighing 25 lbs., is hung up by strings AC and BD, passing round smooth pulleys at C and D, whose distance apart is greater than AB. From the string AC hangs a weight of 23 lbs. What weight must be suspended from the other string to maintain the rod in a position inclined at 30° to the horizontal? Find also the inclinations of the strings to the vertical.

28. A uniform beam, length $2a$, rests with one end on the ground (which is rough) and across a marble block, height h, which is smooth.

When the beam makes an angle α with the horizontal, find the force of friction F and the normal reaction N.

Plot the numerical value of the ratio $\frac{F}{N}$ taking $a = 5$, $h = 3$, for a series of values of α. Hence find the greatest possible inclination of the beam, if the coefficient of friction is $\frac{3}{4}$.

29. The centre of gravity G of a disc of 2 feet radius, weighing 10 pounds, is 6 inches from its centre O. The disc stands on a rough horizontal plane with OG horizontal, being kept in position by a horizontal force applied at the highest point. Find this force, and the force of friction.

What is the coefficient of friction if the disc is on the point of slipping?

30. In a balance the arms are 12 cms. long, the weight of the beam is 200 grs., and the weight of each scale-pan 65 grs. The C.G. of the beam is 2 mm. below the point of support. Each pan is hung so that the C.G. of the pan whether loaded or not is vertically below the point of support of the pan, and the points of support of the beam and pans are in a straight line. Find the angular deflection of the beam when the load in one pan is 50 grs., and in the other 50·1 grs. [Army.]

31. AB, CD are two equal uniform rods which can turn freely about the two points A and C respectively, the line AC being horizontal. To B and D are fastened the ends of a weightless string which passes through a ring R whose weight is half that of either of the rods AB, CD. Show that, if in the position of equilibrium AB and BR make angles and α and β respectively with the vertical, then $\tan\beta = 3\tan\alpha$. Find also the tension of the string. [Army.]

32. A heavy uniform rod weighing 20 ozs., freely jointed at one end, is kept in a horizontal position, partly by a force of 18 ozs. at the other end whose direction makes an angle of 30° with the rod, partly by a vertical force at the centre. Find the vertical force. Also find the magnitude and direction of the strain on the hinge.

33. AB and BC are two equal uniform beams freely hinged at B. Each beam weighs 70 lbs. and is 6 feet long. The ends A and C are freely hinged to two fixed points situated in the same horizontal line 10 feet apart. The system rests with ABC in a vertical plane. Find the magnitudes and directions of the reactions at A, B and C.

34. Two bodies connected by a string passing over a smooth pulley

touch each other at one point; show that the pressure between them cannot act horizontally unless their weights are equal.

35. A uniform heavy rod AB can move freely about the extremity A, and has attached to the other extremity a string, which, passing over a smooth pulley vertically above A and at a distance from it equal in length to AB, sustains a body whose whole weight is $\frac{1}{2}$ that of the rod; find the inclination of the rod to the vertical in the position of equilibrium.

In some problems involving friction, slipping can take place in more than one way.

If the forces tending to produce movement are applied very gradually we can determine by the rules of statics the mode in which motion will commence. If the forces are applied suddenly, quite other considerations require attention and the problem becomes a dynamical one.

36. A thin uniform rod with one edge hinged to a table rests over a cylinder, radius r, which lies on the table, the rod being at right angles to the axis of the cylinder. A gradually increasing horizontal force is applied at the centre of the cylinder, perpendicular to its axis. Will the cylinder slip on the table and roll under the rod, or vice versâ, supposing rod and cylinder equally rough?

The given dimensions and the forces acting are shown in the figure. Writing down the equations of equilibrium for the cylinder we have—

Moments about centre $\qquad F_0 - F_1 = 0,$

Moments about hinge

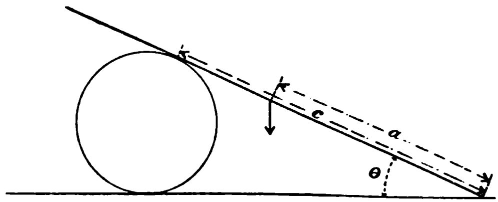
$$Pr + W_1 r \cot\frac{\theta}{2} - N_0 r \cot\frac{\theta}{2} + N_1 r \cot\frac{\theta}{2} = 0,$$

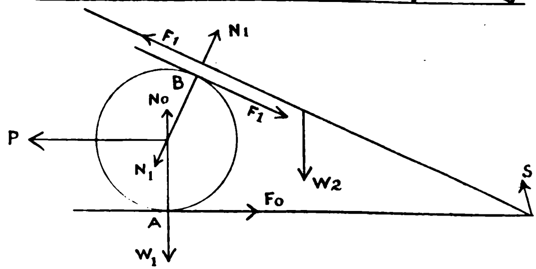
or
$$N_0 = N_1 + W_1 + P \tan\frac{\theta}{2}.$$

Hence N_0 is greater than N_1, and therefore—

$$\frac{F_0}{N_0} < \frac{F_1}{N_1},$$

therefore $\frac{F_1}{N_1}$ will reach its limiting value before $\frac{F_0}{N_0}$ does





so; that is, the cylinder will roll on the table, and slip from under the rod.

The case when rod and table are not equally rough may be left to the student, who should take some numerical instances.

Examples.

37. A rod AB, weighted so that its C.G. is not at its middle point, rests horizontally on a fixed support at one end, and on a cylindrical block at the other end, which block rests on a table. Suppose a very slowly increasing horizontal force to be applied to the centre of the block. Find whether the block will slip on the table or will roll on the table and slip from under the rod, or will

pull the rod off its support. Assume the support, block and table to be equally rough.

38. Work out the preceding example for the case of a uniform rod —(*a*) When the pulling force is applied at a height from the ground is equal to $\frac{3}{4}$ of the diameter; (*b*) when the roller is midway between the centre and end of the rod.

Method of Sections.—If a body is in equilibrium every portion of it is in equilibrium, and we may, if we please, consider the forces acting upon any portion of the body which we can distinguish from the rest.[1]

For example, consider a uniform heavy rod, weight W, hung up by one end. The tension in the rod is greatest near the top and very small near the bottom.

To find the tension, say, two-thirds of the way down, consider the lower third of the rod. The forces acting on it are its own weight $\frac{W}{3}$, and the pull of the upper part on it, say T. These forces being in equilibrium are equal and opposite, whence $T = \frac{W}{3}$.

39. Give a formula for the tension at a distance x from the lower end, the rod being of length a and weight W.

40. One end of a uniform chain that weighs 25 lbs. is attached to a hook, and the chain hanging vertically down supports a weight of 20 lbs. attached to the lower end. What are the tensions of the chain at each end and the middle?

41. A heavy chain of length 8 feet 9 inches, weighing 15 lbs., has a weight of 7 lbs. attached to one end, and hangs in equilibrium over a smooth peg. What length of it hangs on either side?

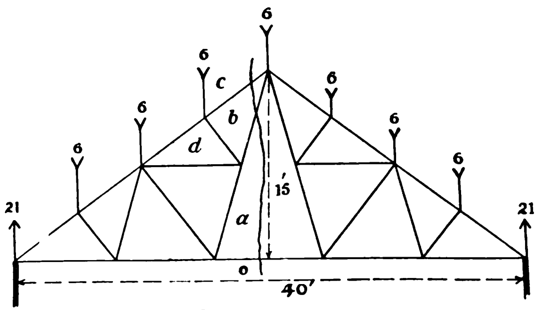
42. A bar of iron 2 feet long, $1\frac{1}{2}$ inches by $\frac{1}{4}$ inch, is suspended vertically. Find the tension at distances 6, 12, and 18 inches from the lower end.

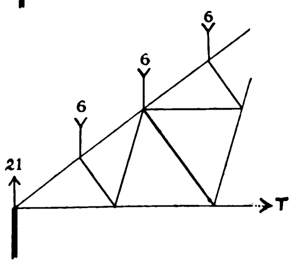
---

[1] We have already given an instance of this method in Chapter IX., on the suspension bridge.

43. A vertical iron rod 60 feet long tapers uniformly from diameter 4 inches to diameter 2 inches, and carries a weight equal to its own weight at its lower end. Find the tensions at every 10 feet, and construct a graph showing the relation between $\dfrac{\text{tension}}{\text{area of cross section}}$ and distance from lower end. Which end should be uppermost?

We can sometimes find the stress in a particular bar of a frame very simply by considering the equilibrium of a part of the frame. This is especially useful if we only require the stress in this particular bar and do not wish to work out the complete graphical solution.





A roof—sometimes called the "Belgian" roof—is shown with the forces in hundredweights acting on it in the figure. Required the tension in the bar *oa*. Imagine a line drawn,

say, a little to the left of the apex, and consider the equilibrium of the portion of the roof to the left of this line.

The forces acting are: (1) The loads as shown in the figure; (2) the upward pressure of the left-hand support (assumed vertical), the magnitude of this pressure is 21 cwts.; (3) the action of the right-hand portion of the roof, this consists of the forces in the three bars, *ab*, *bc* and *ao*.

The student may ask why the force in some other bar, such as *bd*, is not included. If he will try to state in which direction the force in *bd* urges the portion of the roof shown in the lower figure *as a whole*, he will probably be able to answer the question for himself. Noticing that the lines of action of the forces in *ab* and *bc* each pass through the apex, we see that we have only to take moments about the apex to find T. In fact—

$$T \times 15 + 6 \times 5 + 6 \times 10 + 6 \times 15 - 21 \times 20 = 0,$$
$$T + 2 + 4 + 6 - 28 = 0,$$
$$T = 16 \text{ cwts.}$$

As T comes out positive its direction really is as indicated in the diagram and as assumed in the moment equation.

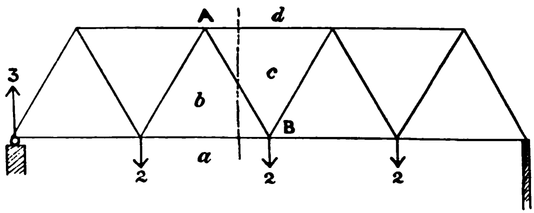
EXAMPLE.

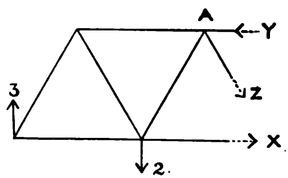
44. Is the bar *oa* a strut or a tie?

In the preceding examples we have applied the method of sections to finding the force in a single bar of a frame. If we can draw a section which does not cut more than

three bars, the forces in these three bars may be calculated by writing down the equations of equilibrium for the forces acting on the portion of the frame on one side of the section.

Thus, in the Warren girder, shown in the upper





diagram, consisting of equilateral triangles of side $2a$, consider the portion to the left of a section through the three bars ab, bc, cd. The forces acting on it (in tons) are shown in the figure, and we have

Resolving vertically, $3 - 2 - Z \sin 60° = 0$,

whence $$Z = \frac{2}{\sqrt{3}} = 1{\cdot}155 \text{ tons.}$$

Moments about A,

$$3 \times 3a - 2 \times a - X\, 2a \sin 60° = 0,$$

$$\therefore\ X = \frac{7}{\sqrt{3}} = 4{\cdot}041 \text{ tons.}$$

Resolve horizontally, $Y - X - Z \cos 60^\circ = 0$,

$$\therefore\ Y = X + \tfrac{1}{2}Z,$$
$$= 4{\cdot}041 + 0{\cdot}577,$$
$$= 4{\cdot}618 \text{ tons.}$$

Remarks.—The true direction of Z, since its value comes out +, is the same as that assumed for it. If we had assumed Z to act in the opposite direction we should have found a negative value $-\ 1{\cdot}155$ for it.

Examples.

45. Show that Y could also be found by taking moments about B.

46. In a Warren girder with six divisions in the lower flange, work out by the method of sections the stresses in each bar, supposing that each lower joint is loaded with 2 tons.

47. In the preceding example suppose each top joint loaded with 2 tons also. Work out the stresses.

48. Work out the stresses as a load of 20 tons is placed at each bottom joint in turn of the girder in Example 46.

49. In a Warren girder which is supported at its ends and which bears loads of any magnitudes at the joints of the upper and lower flanges, prove that (1) every bar of the upper flange is a strut; (2) every bar of the lower flange is a tie; (3) the diagonal bars or "braces" are alternately struts and ties counting from the ends of the girder, except at one joint of the upper or lower flange, at which there are two struts or two ties.

If we have any frame, and can find a section which cuts not more than three bars, the stresses in these bars can be found, unless they all three meet in a point.

If now we can draw another section meeting not more than three fresh bars, we can proceed, provided these fresh bars do not meet in a point.

If at any time the only sections we can draw all cut

more than three non-concurrent fresh bars, or more than two concurrent fresh bars, the method fails.

EXAMPLES.

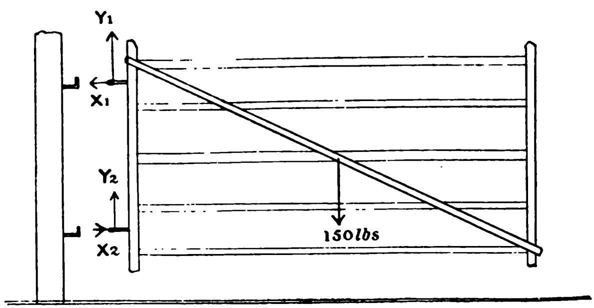
50. Examine the above statements.

51. Why is it necessary that the three fresh bars should not be concurrent?

52. Examine whether it is possible to work out completely the Belgian roof by the method of sections.

Indeterminateness.—The following problem draws our attention to a matter of the utmost practical importance.

A uniform gate, measuring 10 by 3 feet and weighing 150 lbs., is "hung" from a vertical gate-post by two rings fixed to the gate, which fit over two bent pins 2·5 feet



apart in the post. The post and gate are shown separately in the diagram. Required, the reactions at the pins.

Calling X_1, X_2 and Y_1, Y_2 the horizontal and vertical components of the actions of the pins, the forces acting on the gate are shown in the figure.

Resolving horizontally—

$$X_1 - X_2 = 0.$$

Resolving vertically—

$$Y_1 + Y_2 = 150.$$

Taking moments about the lower pin—

$$150 \times 5 = X_1 \times 2{\cdot}5,$$

or

$$X_1 = X_2 = 300 \text{ lbs.}$$
$$Y_1 + Y_2 = 150 \text{ lbs.}$$

We cannot find the values of Y_1 *and* Y_2 *separately.* Y_1 and Y_2 are said to be *indeterminate.*

It is not difficult to see why we cannot calculate Y_1 separately. Suppose the rings on the gate are ever so little too close together. Then the top ring will come right down on its pin, while the lower ring will be off the horizontal portion of the lower pin. Consequently if this were really the case we should have (supposing the pins smooth)—

$$Y_1 = 150, Y_2 = 0,$$

On the other hand, if the rings were a little too far apart, the lower one would press hard down on the pin, and the upper one would not press vertically on the upper pin at all, so that in this case we should have—

$$Y_1 = 0, Y_2 = 150.$$

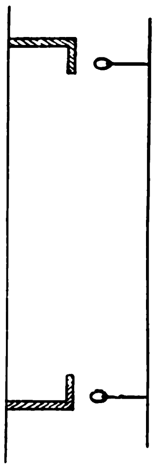
The problem as stated does not enable us to decide how the vertical pressure is distributed between the pins. In any actual case the pressures on the pins are definite.

But opening the gate and shutting it again may alter them.

From a practical point of view indeterminateness is bad. Consider the following cases—

(1) Problem as stated above. We must make *each* pin capable of bearing a vertical thrust of 150 lbs.

(2) Suppose the top pin turned round thus. Then putting friction out of the case, all the weight must be on the lower pin, and one need only make one pin strong enough to bear the vertical thrust of 150 lbs.



In either case the pins must be capable of bearing a horizontal force of 300 lbs.

EXAMPLES.

53. Examine the effect on the reactions of diminishing the distance between the pins to 18 inches, and altering the rings to correspond.

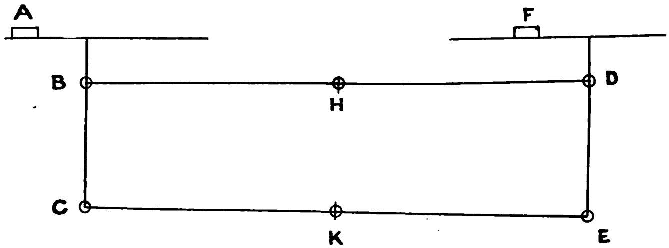
54. Suppose three men carrying a uniform ladder, one at the middle and one at each end. Show that the distribution of weight among the three is indeterminate, and account for this result on common-sense grounds.

55. A beam loaded with any weights rests on supports at A and B in the same horizontal line. Prove that the vertical reactions at A and B are determinate, but the horizontal reactions indeterminate. For this reason when a roof or girder rests on two supports one end is generally placed on rollers, so as to ensure that there can be no horizontal reactions, and to allow room for any expansion or contraction due to changes of temperature (see p. 270).

56. A heavy rod can turn about an axis which is inclined at 10° to the vertical, and turns on hinges 4 inches apart. The rod is 30 inches long and weighs 5 lbs. Calculate the reactions on the hinges as completely as possible.

Work.—The principle of work often enables us to deal easily with a statical problem which would be somewhat difficult if treated otherwise.

An example is furnished by a mechanism known as



Roberval's balance, because it was invented by Gilles Personne de Roberval,[1] which is the foundation of most common weighing machines—such as are used for ordinary household purposes.

Two equal levers BHD, CKE, can turn about their centres H, K, which are fixed. Pieces ABC, FDE, are hinged to the levers at BC and DE respectively—

and $$BC = HK = DE.$$

Notice that BC and DE are always vertical whatever may be the inclination of the levers BD and CE.

EXAMPLE.

57. If F is pushed down 1 inch how far will A rise ?

[1] Born at Roberval in 1602, published a treatise on mechanics in 1636, died at Paris in 1675. The story is related of him, which has since been told of many mathematicians, that his only comment, on hearing a tragedy read was, "What does it prove?"

Suppose a weight P placed at any point F on one platform, balances a weight W at any point A on the other platform, P descending a distance a can do an amount Pa of work.

Work necessary to raise W through a distance $a = Wa$.

But since the weights balance, then if we neglect all sources of loss of work, the very smallest possible excess of effort will move the balance. Hence, in the ideal case—

$$Pa = Wa,$$

$$\therefore\ P = W,$$

or equal weights will balance on a Roberval's balance, at whatever point on the platforms they may be placed. This appears at first sight paradoxical. To explain the paradox, note that if we release the point K, equilibrium will be destroyed.

Example.

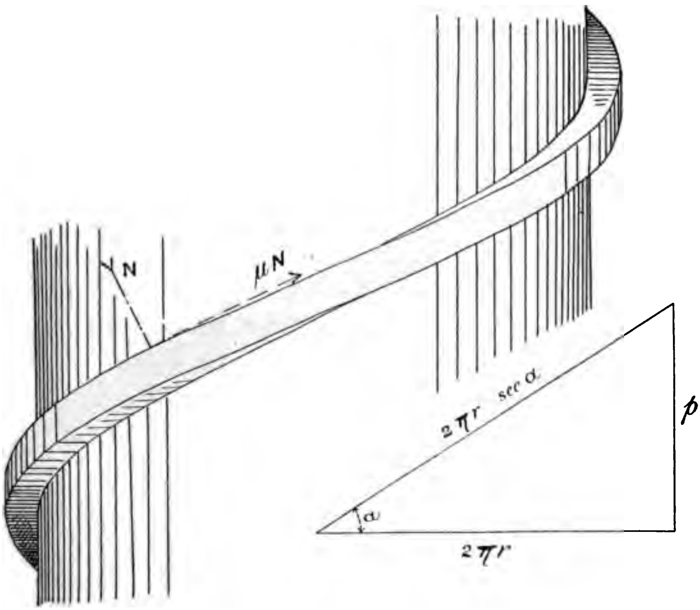
58. Find the horizontal pressure of the pin K if equal weights of 10 lbs. are placed on the balance, one 2 inches further from HK than the other, and if HK = 4 inches.

Rough Screw.—As a second illustration we will find the thrust which can be exerted by a rough screw working in a fixed nut—when a force P is applied to the screw by a handle of length l.

In a rough screw the efficiency depends on the coefficient of friction, and on the shape of the screw thread.

Suppose the thread rectangular, that is, let it be traced out by a small rectangle moving round a cylinder,

and at the same time advancing parallel to the axis. A look at a rectangular screw thread will be the best means of forming a clear notion of its shape.



Considering any small portion of the surface of contact between the screw and the nut, let there be a normal reaction N, and a force of friction μN. N makes an angle a, with the axis of the screw, and μN an angle $90 - a$. These angles are the same for all the series of forces such as N and μN, and hence, resolving vertically for the forces acting on the screw—

$$-X + (\Sigma N)\cos a - \mu \sin a \Sigma N = 0,$$

or
$$\Sigma N = \frac{X}{\cos a - \mu \sin a},$$
$$= \frac{X \cos \lambda}{\cos (a + \lambda)},$$

if $\mu = \tan \lambda$.

Now the work done by P in one revolution is $2\pi l P$, and this is expended in exerting a thrust X through a distance $p = 2\pi r \tan a$, and in pushing a distance $2\pi r \sec a$ (see figure) against the forces of friction. That is, dividing by 2π—

$$Pl = Xr \tan a + \mu(\Sigma N).r \sec a,$$

or
$$Pl = Xr \tan a \left[1 + \frac{\tan \lambda}{\sin a}.\frac{\Sigma(N)}{X}\right],$$
$$= Xr \tan a \left[1 + \frac{\tan \lambda}{\sin a}.\frac{\cos \lambda}{\cos \overline{a + \lambda}}\right],$$
$$= Xr \tan a \left[1 + \frac{\sin \lambda}{\sin a \cos \overline{a + \lambda}}\right].$$

This can be further simplified by writing—

$$\sin \lambda = \sin (a + \lambda - a) = \sin \overline{a + \lambda} \cos a - \cos \overline{a + \lambda} \sin a.$$

We thus obtain—

$$Pl = Xr \tan a . \frac{\sin \overline{a + \lambda} \cos a}{\sin a \cos \overline{a + \lambda}},$$

and the efficiency is therefore

$$\frac{Xr \tan a}{Pl} = \frac{\tan a}{\tan (a + \lambda)}.$$

Notice that we have assumed that the distance from the axis at which the forces of friction act is r.

Example.

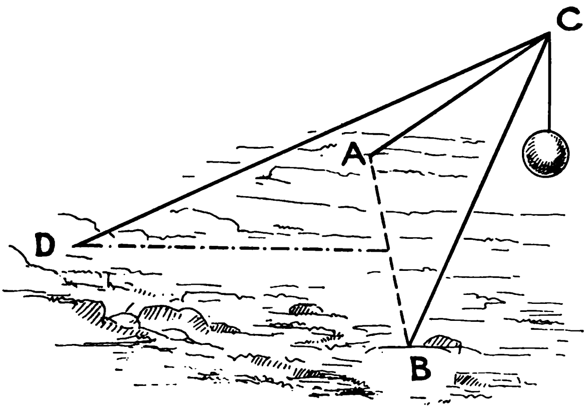
59. Find the condition that a pressure W applied parallel to the axis of a rough rectangular screw may fail to move it.

CHAPTER XI

FORCES IN SPACE

IN this chapter we shall consider some cases in which the forces acting in a body are not all in one plane.

Sheers.—In some cases the system of forces is very readily reduced to a set of coplanar forces. An instance of this is furnished by the sheers, a contrivance used for raising and moving heavy weights.



AC, BC are two equal stout straight rafters or beams called the legs of the sheers. The feet A, B are secured to the ground, but loosely so that, though they cannot slip,

yet the legs can turn about the line AB. AC, BC are lashed together at C. A hook from C carries the load. A backstay, or rope CD, in the vertical plane bisecting AB at right angles is carried from C. By hauling in or letting out this rope the weight can be shifted backwards or forwards.

Large sheers were used in the old days for putting the masts into a ship, and they are still used for similar purposes, *e. g.* for putting the funnel into a steamer.

The forces acting at C are—(1) the load W; (2) the thrust of the legs, each, by symmetry, R, acting along AC and BC; (3) the tension T of the backstay. It is evident that these forces are not coplanar.

We can in the present case evade this difficulty by noticing that the resultant, say S, of the two thrusts R and R acts, by symmetry, along the bisector of the angle ACB, and that the forces S, T and W *are* in one plane.

EXAMPLES.

1. Given AB = 15 feet, AC = BC = 30 feet, CD = 50 feet, distance of D behind the line AB = 33 feet, W = 4 tons; find R and T.

[The steps in the work will be as follows—(1) Draw the triangle ABC in its true shape. Bisect AB in M, measure CM. Check your result by calculating CM. (2) Draw the triangle DMC in its true shape. (3) The forces acting at C in the plane DMC are: W vertical, T along CD, S along CM. Draw a triangle of forces, and determine T and S. (4) Returning to the triangle ABC, S acting along CM is the resultant of R along CA and R along CB. Find R, using the value determined in (3) for S.]

2. In the preceding example find the "overhang," *i. e.* find how much the foot of the perpendicular from C on the ground is in front of AB.

3. How much rope must be paid out to increase the overhang by 3 feet, and what will be then the tension in the backstay?

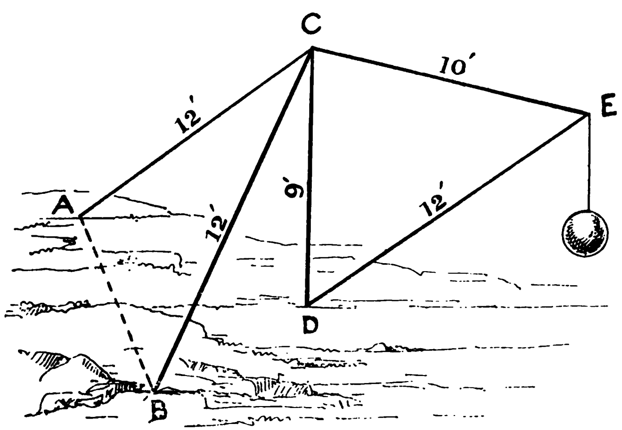
4. The legs of a pair of sheers are each 40 feet long, and the backstay 55 feet long. The feet of the legs are 18 feet apart and the foot of the backstay 25 feet behind the line joining the feet of the sheers legs. Find the tension in the backstay and the thrusts in the legs under a load of 10 tons.

5. Find in the previous example the angle which the plane of the sheers legs makes with the horizontal, and the angles the sheers legs make with the horizontal.

6. The backstay in the sheers described in Example 4 cannot safely bear a tension exceeding 13 tons. Find the limit to the length of the backstay.

7. Length of legs, 15 feet; distance between feet, 16 feet; length of backstay, 25 feet; distance of foot of backstay behind the line joining the feet of the legs, 19 feet. How much must the foot of the backstay be shifted forward to move the load 3 feet forward, and how much will this movement lower the weight?

8. A derrick is shown in the figure with its dimensions. Find



the stresses in the various bars, supposed CD can turn freely about the lower end D, and that the plane CDE bisects AB. Examine the general effect of building CD firmly into the ground. AB = 6 feet.

If in the derrick shown above the plane CDE cuts the line AB in any point F, we can find the stresses in the various bars by noticing that the resultant, say R, of the stresses in the backstays AC and BC must act along the line CF. To prove this, consider the forces acting on the joint C. The resultant R′ of the stresses in CE and CD lies in the plane CDE. The resultant R of the stresses in CA and CB lies in the plane CAB. Now R and R′ must be equal and opposite, and consequently must act in the same line, which can only be the line CF common to the planes CDE and CAB.

Examples.

9. Find the stresses in the bars of Example 8 if AF = 2 feet, BF = 4 feet.

10. A wheel whose rim is semi-circular runs in a smooth horizontal V-shaped groove. If the load on the wheel is 50 lbs., find the reactions of the groove if the sides are each inclined at (1) 60°, (2) 40°, to the horizontal.

11. A small V-shaped trough has its sides equally inclined and its edges inclined at 20° to the horizontal. A marble is placed in the trough resting against the sides and against one end. Find the pressure on the sides and end if the marble weighs 1 ounce and the angle of the V is 35°.

12. Two marbles, radii $\frac{1}{2}$ inch and 1 inch, are placed in the trough of Example 11, the smaller one resting against the end. Find the pressure between the two, supposing their weights to be $\frac{1}{4}$ and 2 ozs. respectively.

13. A vertical post carries two wires, tension 100 lbs., leaving it horizontally at 40 feet from the ground, and making 145° with one another. To what tension should a stay be brought, in order that there may be no lateral force on the post, supposing the stay fastened at the same height as the wires, and to be 55 feet long?

14. A uniform rectangular platform ABCD is to be slung in a horizontal position by vertical ropes attached to A and B, and by a

third rope. Find the direction of this third rope and its point of attachment if its tension is to be double that of the others.

15. Three equal rods OA, OB, OC, each 8 feet long, are tied together at O from which 2 cwts. are suspended. The feet A, B, C of the rods form a horizontal equilateral triangle, side 5 feet. Find the thrust in each rod.

16. Telegraph poles 60 yards apart are spaced along a curve of 400 yards radius, a wire is stretched in an approximately horizontal line from pole to pole, the tension being 1200 lbs. Calculate the horizontal pull on each post, and if each pole is relieved of this pull by a wire stay inclined at 30° to the vertical find the tension in the stay.

17. Two men pull at a tree which has been partly sawn through, using ropes each 30 feet long attached to a point in the trunk 10 feet from the ground. The men are 24 feet apart and pull with a force of 80 lbs. each. Find the resultant force on the tree.

Three Concurrent Forces.—The proposition of page 87 may readily be extended to forces in three dimensions.

Thus let OA, OB, OC, be three lines along which forces P, Q and R act represented by $l\,.\,\mathrm{OA}$, $m\,.\,\mathrm{OB}$ and $n\,.\,\mathrm{OC}$ respectively where l, m, n are numbers.

Divide BC in H so that—

$$m\,.\,\mathrm{BH} = n\,.\,\mathrm{CH}.$$

By the proposition referred to, the resultant of Q and R is a force represented by $(m + n)\mathrm{OH}$ and acting along OH.

Applying the proposition once more, and dividing AH in G so that

$$l\,.\,\mathrm{AG} = (m + n)\mathrm{GH},$$

the resultant of P, Q and R is a force represented by $(l + m + n)\mathrm{OG}$ and acting along OG.

Examples.

18. Show that if a weight W is supported by three spars AO, BO, CO, and if the vertical through O meets the plane of ABC in G, AG produced meets BC in H, then the thrust in the bar OA is $\frac{HG \cdot OA}{AH \cdot OG} \cdot W$.

[As has already been observed in the case in which the forces are coplanar, this formula has the advantage of involving only lengths which can be measured from the object itself.]

19. A weight of 300 lbs. is supported by three bars OA, OB, OC, whose feet rest in an inclined plane. OA = 10·5 feet, OB = 7·9 feet, OC = 9·1 feet. The vertical through O meets the plane in G and it is found that OG = 8·5 feet, AG = 3·5 feet, BG = 3·9 feet, AB = 6·7 feet, BC = 7·35 feet, CA = 11·35 feet. Apply the method of the preceding example to find the stresses in the three bars.

Experiment 1.

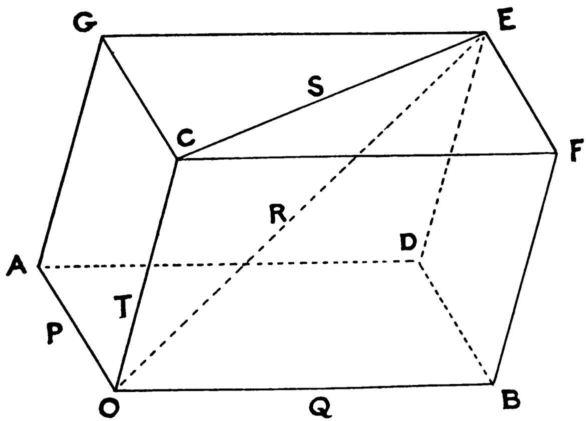
Support a weight by three strings, not all in one plane, passing over pulleys and counterpoised by pieces of lead. Take the necessary measurements and apply the preceding formulæ to find the tensions in the strings. Weigh the pieces of lead and compare with the results of your calculations.

General Propositions.—As soon as it is realised that a force is a "vector" quantity which can be adequately represented by a straight line, it is evident that many of the propositions which have been stated of forces in one plane may be at once extended to forces in space. For instance—

(1) The resultant of three forces P, Q, T acting at a

point is represented by the diagonal of the parallelopiped whose three adjacent edges represent the forces. For the diagonal OE represents the resultant of T and S—S, the force represented by OD or CE, being itself the resultant of P and Q.

(2) A force acting at a given point may be resolved into components in any three given lines through that point (not all in one plane), and there is only one way of doing this. For there is only one parallelopiped with a given

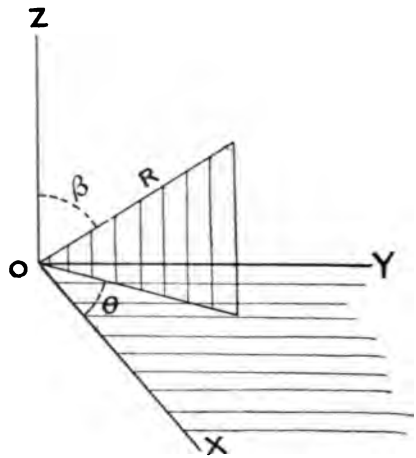


diagonal OD and its adjacent edges along given lines OA, OB and OC.

EXAMPLES.

20. A vertical plane makes an angle θ with a vertical plane running E. and W. In the first plane a force R acts, at an angle β

to the vertical. Find the components of the force in the three directions, (1) E. and W., (2) N. and S., (3) vertical.



[Calling these X, Y, Z, we obtain—

$$Z = R \cos \beta,$$
$$X = R \sin \beta \cos \theta,$$
$$Y = R \sin \beta \sin \theta.]$$

21. A kite drags a boat along. The tension of the string is 25 lbs. The string is inclined at 25° to the horizontal and the vertical plane containing it makes an angle of 17° with the direction of the boat. Find the components tending to pull the boat along, to give it leeway, and to lift it bodily up.

22. Draw the face OCBF (Fig. p. 333) and let its diagonals meet in M. Show that—

$$CE^2 + EB^2 = 2CM^2 + 2EM^2 \quad \ldots \ldots \quad (1)$$

and

$$OE^2 + EF^2 = 2OM^2 + 2EM^2 \quad \ldots \ldots \quad (2)$$

Show by subtraction that—

$$OE^2 + EF^2 - CE^2 - EB^2 = 2(OM^2 - CM^2),$$
$$= \tfrac{1}{2}(BC^2 - OF^2).$$

23. If the forces are P, Q, T, and $\angle AOB = \alpha$, $\angle BOC = \beta$, $\angle COA = \gamma$, show, by the preceding example, that—

$$R^2 = P^2 + Q^2 + T^2 + 2PQ \cos \alpha + 2QT \cos \beta + 2PT \cos \gamma.$$

24. If $P = 10$ lbs., $Q = 12$ lbs., $T = 14$ lbs. and $\alpha = \beta = \gamma = 100°$, find R.

25. Three forces, P, Q, T, act along three lines OA, OB, OC. Find the forces acting along the bisectors of the angles AOB, BOC, COA which have the same resultant.

[Hint: Let $P = X + Y$, $Q = Y + Z$, $T = Z + X$.]

Couples in Space.—Graphical representation of a couple. The characteristic features of a couple are—

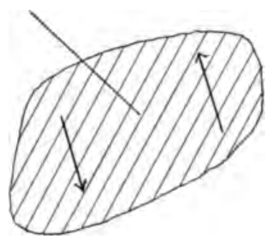
(i) Its moment.

(ii) The "orientation" of its plane, that is, the direction of its plane.

(iii) The sense in which it tends to produce rotation.

A straight line can represent, upon any given scale, the moment of the couple or the "torque" due to the couple.

If the straight line is drawn perpendicular to the plane of the couple it indicates the "orientation" or direction of the plane. Thus a vertical line would represent a couple in a horizontal plane. A horizontal line due S would represent a couple in a vertical plane running E and W.



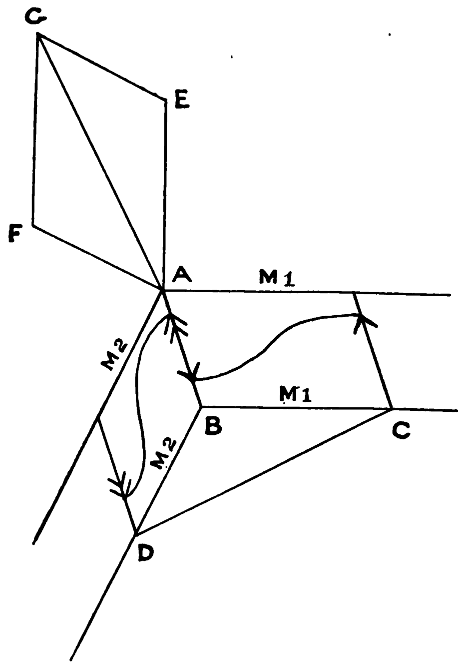
Suppose the "sense" of the couple to appear counter-clockwise in its plane when viewed by an observer. Let us adopt the convention that the line representing the couple shall be drawn from the plane *towards* the observer in this case, and away from the observer if the sense of the couple appears clockwise.

On this convention a line can be drawn to represent the couple in magnitude, sense and orientation.

EXAMPLE.

26. Show that on whichever side of the plane the observer stands he will obtain the same representation of the couple by the above method. Verify by drawing parallel lines to represent the forces of the couple on a piece of glass.

Composition of Two Couples whose Planes are not Parallel, by Geometric Addition.—Let the moments of the couples



be M_1 and M_2. Let the planes intersect in the line AB. The couple M_1 may be furnished by a force of 1 lb.

acting along AB, and an equal force in a line parallel to AB at a distance BC = M_1 from it (see p. 120).

In like manner the second couple may be furnished by a force of 1 lb. along BA, and an equal force in a line parallel to BA at a distance BD = M_2. The figure is drawn for the case in which one plane is horizontal and in which both couples appear to turn counter-clockwise to an observer looking down from above the planes. Evidently the resultant is a couple whose moment is CD.

The first couple is represented by a line, say AE, of length M_1 drawn at right angles to the first plane. The second couple is in like manner represented by a line AF, of length M_2, drawn at right angles to the second plane.

In the figure the angles ABC, ABD are right angles, and hence the planes DBC, AFE are parallel.

The resultant couple is represented by the diagonal AG of the parallelogram whose adjacent sides are AE and AF, for the triangles BCD and AEG are congruent, having—

$$AE = BC,$$
$$EG = BD,$$
and
$$\angle GEA = 180° - \angle EAF = \angle CBD,$$

and consequently AG is equal to CD, and is perpendicular to the plane of the resultant couple, that is, AG represents the resultant couple in magnitude, sense and orientation.

EXAMPLE.

27. Draw the figure, and verify the conclusion, for the case when one of the couples, say M_1, is clockwise and the other counter-clockwise to an observer looking down from above. [Notice that a section by a plane at right angles to AB gives all that is necessary.]

Having now shown that couples are vector quantities which are compounded by the parallelogram law, it is evident that several propositions relating to couples may e at once inferred.

Thus (1) a couple represented by OE may be replaced by three couples represented by the adjacent edges OA, OB, OC of a parallelopiped with OE as a diagonal.

EXAMPLES.

28. The straight line representing a couple is often termed its *axis*. If the axis of a couple of moment G is inclined at an angle β to the vertical and is in a vertical plane, making θ with the E. and W. plane, find the component couples whose axes are E. and W., N. and S., and vertical.

(2) Couples represented by their axes may be resolved into components in any three directions at right angles.

(3) A couple has no tendency to turn a body about a line at right angles to the axis of the couple. Hence the tendency of a couple to rotate a body about a line is measured by the resolved part of the axis of the couple along that line.

29. A body is free to turn about a line in the plane XOY, making an angle α with the E. and W. line. If the couple specified in the preceding question acts on the body, what is the couple tending to revolve the body about the axis?

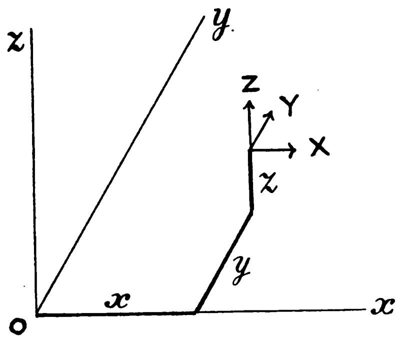
30. A body can turn about a line through O inclined at angles α, β, γ to OX, OY, OZ respectively. A couple G acts on the body whose axis makes angles λ, μ, ν with OX, OY, OZ respectively. What is the component couple tending to cause the body to revolve about the line?

[$G(\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\lambda\cos\nu)$.]

Reduction of Forces in Space.—*Proposition.*—A force

acting at any point in space on a rigid body may be replaced by an equal force at any other point and a certain couple.

Let the force act at the point P whose co-ordinates are x, y, z. Replace it by its three components X, Y, Z



parallel to Ox, Oy, Oz. Introduce at the origin O six forces—two each X, acting in opposite directions along Ox, and similarly for Y and Z. These are in equilibrium of themselves.

The force X at P, and one of the forces X at O form a couple, whose moment about the axis Oz is Xy in a clockwise direction. According to our convention, we shall consider this as a negative moment, $-Xy$. The moment of the Y couple about Oz is $+Yx$, and the moment of the Z couple about Oz is 0. Thus the total moment about Oz is $Yx - Xy$.

Similar expressions can be written down for the moments about Ox and Oy. Hence the original force is equivalent to three forces X, Y, Z acting at O, and three

couples, say L, M, N, whose axes are along Ox, Oy, Oz, where—

$$\begin{aligned} L &= Zy - Yz, \\ M &= Xz - Zx, \\ N &= Yx - Xy. \end{aligned}$$

The resultant of the three couples L, M, N is a single couple, which must be identical with the couple consisting of the original force, P, and an equal and opposite force at the origin.

Conditions of Equilibrium of any Number of Forces.—Replacing each force by its components X, Y, Z at the origin, together with the component couples L, M, N, the whole set of forces is equivalent to

(1) a component force, $\Sigma(X)$ along Ox,
(2) „ „ $\Sigma(Y)$ „ Oy,
(3) „ „ $\Sigma(Z)$ „ Oz,

together with

(4) a component couple, $\Sigma(Zy - Yz)$, axis along Ox,
(5) „ „ $\Sigma(Xz - Zx)$, „ Oy,
(6) „ „ $\Sigma(Yx - Xy)$, „ Oz.

For equilibrium we must have no resultant force, and no resultant couple—

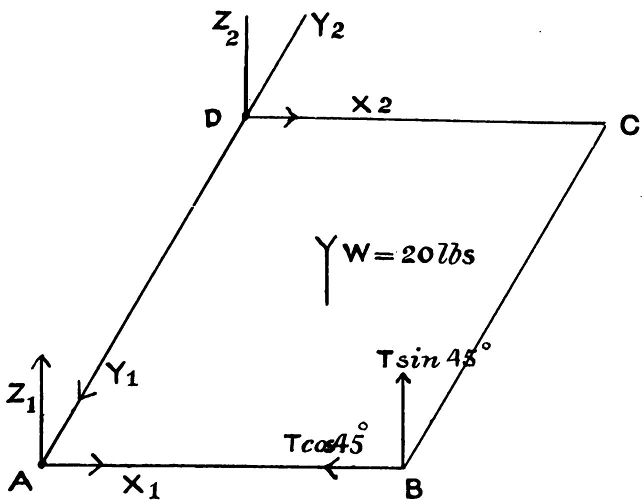
$$\begin{aligned} \therefore \quad \Sigma(X) &= 0, \\ \Sigma(Y) &= 0, \\ \Sigma(Z) &= 0, \\ \Sigma(Zy - Yz) &= 0, \\ \Sigma(Xz - Zx) &= 0, \\ \Sigma(Yx - Xy) &= 0. \end{aligned}$$

In nearly all cases in practice it will be best to choose three rectangular axes, and replace each force by its components parallel to these axes. The equations of equilibrium may thus be readily written down.

EXAMPLE.

31. A horizontal rectangular trap-door ABCD, weight 20 lbs., can turn about hinges at A and D, and is kept in position by a chain attached to B fastened to a point vertically over A, and inclined at 45° to the horizontal. Find the tension of the chain and the reactions at the hinges, if AB = 18 inches, AD = 24 inches.

Take A as origin, axes AB, AD, and vertically



upwards. Replace T by T sin 45° vertical, T cos 45° along BA.

Let X_1, Y_1, Z_1 and X_2, Y_2, Z_2 be the components of the reactions at the hinges. Writing down our six equations, we obtain—

$$X_1 + X_2 - T\cos 45^\circ = 0, \quad . \quad . \quad . \quad (1)$$
$$Y_1 + Y_2 = 0, \quad . \quad . \quad . \quad (2)$$
$$Z_1 + Z_2 + T\sin 45^\circ - 20 = 0, \quad . \quad . \quad . \quad (3)$$
$$Z_2 \times 24 - 20 \times 12 = 0, \quad . \quad . \quad . \quad (4)$$
$$20 \times 9 - T\sin 45^\circ \times 18 = 0, \quad . \quad . \quad . \quad (5)$$
$$X_2 \times 24 = 0, \quad . \quad . \quad . \quad (6)$$

and from these equations we see that—

$$\begin{array}{ll}
(6) & X_2 = 0, \\
(4) & Z_2 = 10, \\
(5) & T\cos 45^\circ = 10, \text{ whence } T = 14{\cdot}1 \text{ lbs.} \\
(1) & X_1 = 10, \\
(3) & Z_1 = 0.
\end{array}$$

We cannot find the *separate* values of Y_1 and Y_2, and to this extent the problem is indeterminate. If the door has a little "play" in the hinges, and the hinges are smooth and well-oiled, $Y_1 = Y_2 = 0$.

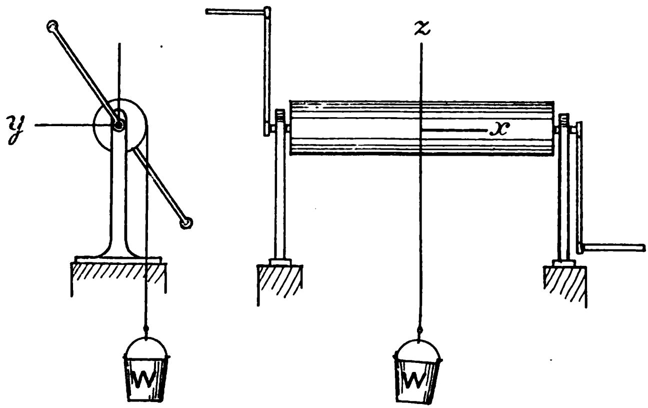
Examples.

32. If in the preceding examples the hinges were 6 inches from A and D respectively, find the tension in the chain and the reactions at the hinges.

33. A bracket is supported by two nails, A, B, and its lower end rests against the wall at C. AB is horizontal = 10 inches, AC = BC = 12 inches. A weight of 20 lbs. rests on the bracket 8 inches

from the wall, and 1 inch from A measured parallel to AB. Find the reactions at A, B and C.

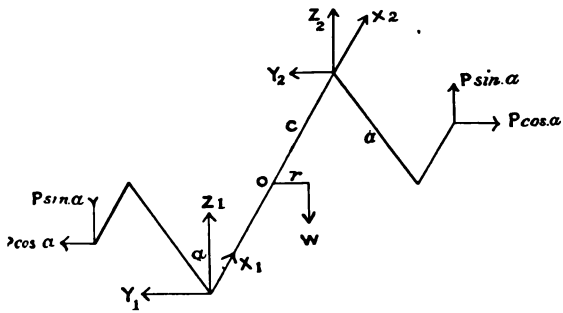
34. A windlass is worked by two handles as shown in the figure.



Take the origin at the middle of the barrel Ox along the axle, Oz vertical, Oy horizontal and at right angles to Ox.

Let r be the radius of the barrel, and $2c$ its length,
l the length of each handle,
a the radius of each handle arm,
the weight act midway as shown,
X_1, Y_1, Z_1 and X_2, Y_2, Z_2 be the components of the reactions at the supports of the axle,
equal forces each P be applied tangentially at the ends of the handles,
the weight of the barrel and handles be neglected.

The system of forces is shown below. Writing down the six equations, we have—



$$X_1 + X_2 = 0,$$
$$Y_1 + Y_2 + P\cos a - P\cos a = 0,$$
$$Z_1 + Z_2 + P\sin a - P\sin a - W = 0,$$
$$2Pa - Wr = 0,$$
$$2P\sin a\,(l + c) + (Z_2 - Z_1)c = 0,$$
$$2P\cos a\,(l + c) + (Y_1 - Y_2)c = 0.$$

From these equations we see that

$$Y_2 = -Y_1 = P\cos a\,.\frac{l+c}{c}$$

$$Z_1 + Z_2 = W,$$
$$Z_1 - Z_2 = 2P\sin a\,.\frac{l+c}{c}$$

whence
$$Z_1 = \frac{W}{2} + P\sin a\,.\frac{l+c}{c},$$
$$Z_2 = \frac{W}{2} - P\sin a\,.\frac{l+c}{c},$$

We cannot find the separate values of X_1 and X_2.

The example illustrates a case of want of balance in a mechanism. It will be seen that as a goes through the series of values from 0° to 360°, Z_1, for instance, varies in magnitude from $\frac{W}{2} + P\frac{l+c}{c}$ as a maximum to $\frac{W}{2} - P\frac{l+c}{c}$ as a minimum and Y_1, Y_2 and Z_2 go through similar periodic changes. The result is to set up a wrenching action at the supports, tending to loosen these and to make the machine rock in its bearings.

EXAMPLES.

35. In an engineer's yard a horizontal rectangular frame, 50 feet long and 28 feet wide, is borne by four pillars at the corners. A travelling girder, parallel to the short sides, rests on the long sides, and carries a lifting apparatus. When this apparatus lifts 10 tons, being then 12 feet from one end of the travelling girder, which is 18 feet from one end of the frame, find in hundredweights the additional forces at the ends of the girder, and how these are distributed among the pillars.

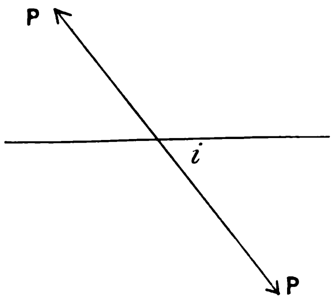
36. A smith's vice stands on three legs whose feet are at the corners of an equilateral triangle whose side is 18 inches. A bolt is held horizontally in the vice parallel to one of the sides of the triangle. A couple of moment 100 lbs.-feet is applied to endeavour to turn a nut which has rusted home on the bolt. Find the reactions at the ground, supposing the total weight to be 250 lbs.

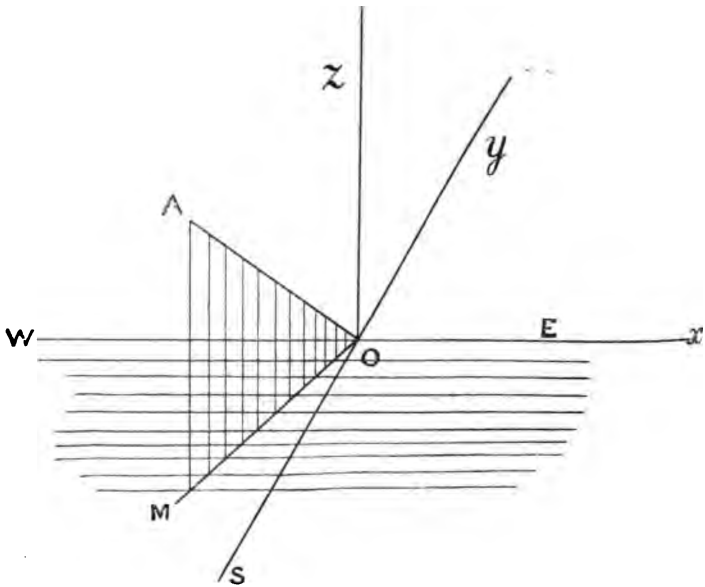
37. In the preceding question what couple applied to the bolt would just begin to upset the vice? [Notice that there are two possible answers depending on the "sense" of the couple.]

38. If in Example 36 instead of a couple a single horizontal force of 100 lbs. is applied to the nut by a vertical spanner 1 foot long, find the vertical reactions at the ground, assuming the C.G. vertically over the centre of the base. [Notice that there are four possible cases, according as the spanner is above or below the nut, and the force acts to the right or left.] Height of nut 3 feet.

Magnetic Needle.—Some additional illustrations are

furnished by a magnetic needle. The effect of the earth's magnetism on such a needle may be represented by two equal and opposite forces supposed to act at the N. and S. seeking poles of the magnet acting toward the N. and S. and at an inclination i to the horizontal. When the needle is in the magnetic meridian and dipping at an inclination i, the lines





of action of these forces coincide, and there is equilibrium. *In any other* position they constitute a couple.

Let AOA′ [A′ is not shown in the figure] represent the magnet,

A and A′ the S. and N. seeking poles,

ZAOM the vertical plane containing the magnet,

and SO the magnetic N. and S. line.

Let $$\angle \text{AOM} = \phi,$$
$$\text{SOM} = a.$$

The magnetic forces acting at A A′ have components—

	A	A′
	$X = O$	O
	$Y = -P\cos i$	$P\cos i$
	$Z = P\sin i$	$-P\sin i$

The co-ordinates of A and A′ are, if AA′ = $2a$—

	A	A′
x	$-a\cos\phi\sin a$	$a\cos\phi\sin a$
y	$-a\cos\phi\cos a$	$a\cos\phi\cos a$
z	$a\sin\phi$	$-a\sin\phi$

Hence the magnetic couple has the three components—

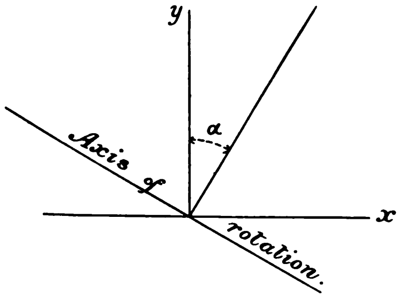
$$\begin{aligned} L &= \Sigma(Zy - Yz) = -2aP[\sin i\cos\phi\cos a - \cos i\sin\phi], \\ M &= \Sigma(Xz - Zx) = 2aP[\sin i\cos\phi\sin a], \\ N &= \Sigma(Yx - Xy) = 2aP[\cos i\cos\phi\sin a]. \end{aligned}$$

Suppose the magnetic needle perfectly free to turn about OZ, and that a couple G, axis along OZ, is applied to turn it. There will be equilibrium when $G - N = 0$, *i. e.* when
$$G = 2aP\cos i\cos\phi\sin a,$$

so that G is proportional to the *sine* of the horizontal

angle through which the needle is deflected out of the magnetic N.S. line.

Next, suppose the needle free to turn about a horizontal



axis, making an angle $90° - a$ with Oy, as shown. The couple tending to turn the needle about this axis is—

$$M \sin a - L \cos a,$$

since couples may be resolved in the same way as forces.

This couple is—

$$2aP[\sin i \cos \phi \sin^2 a + \sin i \cos \phi \cos^2 a - \cos i \sin \phi \cos a]$$
$$= 2aP[\sin i \cos \phi - \cos i \sin \phi \cos a].$$

This couple vanishes when—

$$\sin i \cos \phi = \cos i \sin \phi \cos a,$$

that is—

$$\tan \phi = \tan i \sec a,$$

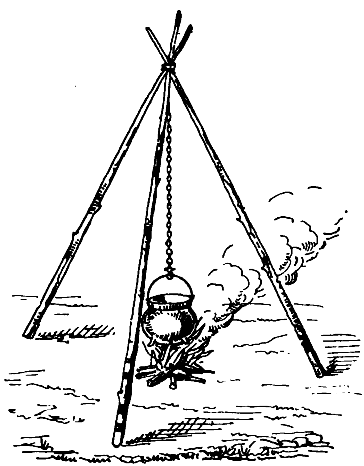
and this equation gives the inclination to the horizontal ϕ at which a magnetic needle will set itself if it is free to turn about a horizontal axis inclined at an angle $90° - a$ to the magnetic north.

EXAMPLES.

39. A circular disc of radius a and weight W rests in a horizontal position suspended by three vertical threads of length l attached to points on its rim at the vertices of an equilateral triangle. Prove that to hold the disc horizontal and twisted through an angle θ a horizontal couple of moment $\dfrac{W a^2 \sin\theta}{\sqrt{l^2 - 4a^2\sin^2\frac{\theta}{2}}}$ is requisite.

40. AC is a line of greatest slope in a plane, inclination i. AB is a smooth fixed rod lying in the plane and making an angle α with AC. Show that a body placed on the plane and resting against the rod will slip unless $\mu > \tan i \cos\alpha$. Assume that the pressure of the rod on the body acts in the inclined plane. [Hint: Reduce the forces to (1) a set acting in the plane, and (2) a set acting at right angles to the plane. *Each* set must be in equilibrium.]

Graphical Treatment of Forces in Space.—We shall consider cases in which the artifice of reducing the forces in action to a system of coplanar forces is not readily applicable. We commence by considering how the object is to be represented.



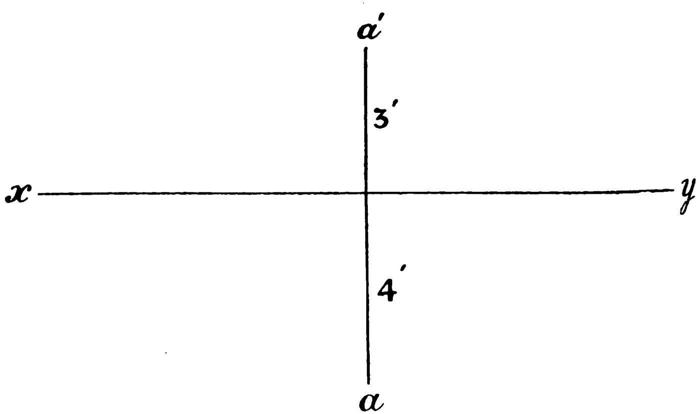
Plan and Elevation.—A freehand or perspective sketch, though conveying a clear idea of the object (see figure), is of no use for taking measurements of lengths.

A *plan* and *elevation*, though less pictorial to the untrained eye, represent many lines of the object in the

true proportions, and furnish a means of constructing the length of *any* line in the object.

In this system a point in space is represented by the feet of the perpendiculars drawn from it to two perpendicular planes—usually horizontal and vertical planes (abbreviated to H.P. and V.P.).

In drawing, the vertical plane is supposed to have been revolved through 90° so as to lie in the horizontal plane.

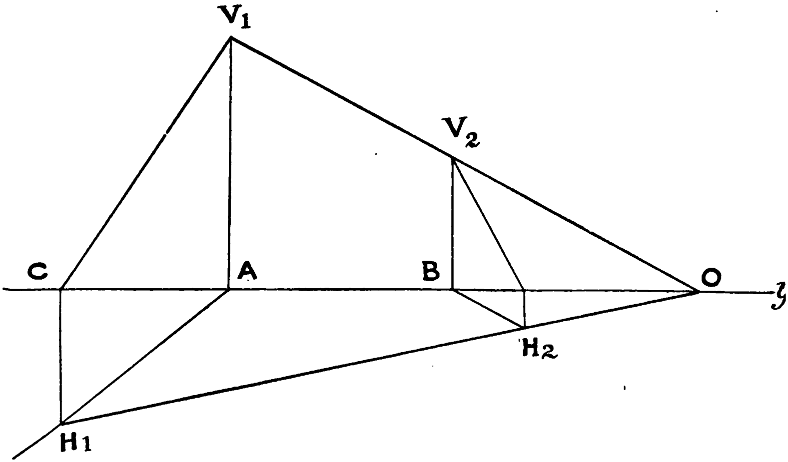


Thus the figure shows on a certain scale the plan, *a*, and elevation, *a'*, of a point 4 feet from one wall and 3 feet from the floor of a room.

A *point* may be indicated by its plan and elevation.

A *straight line* by its plan and elevation, that is, its projections on the planes of reference; or by its *traces*, the points in which it cuts the horizontal and vertical planes.

A *plane* by its traces, the straight lines in which it cuts the horizontal and vertical planes.



In the figure, H_1 V_1 are the traces of a line, H_2 V_2 are the traces of another line.

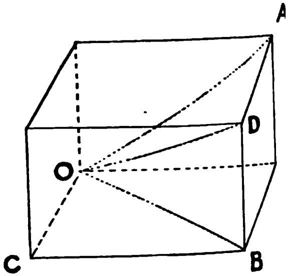
OV_1V_2, OH_1H_2 are the traces of the plane which contains the two lines.

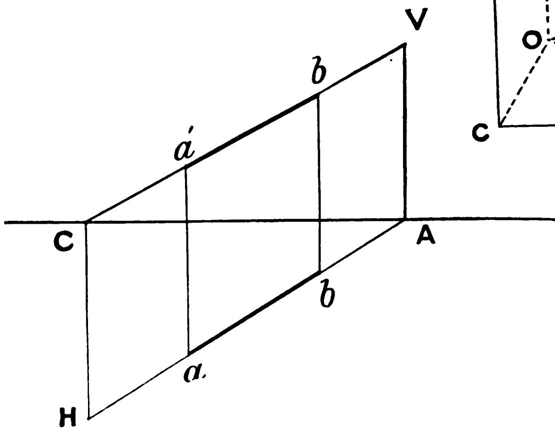
AH_1 is the plan and CV_1 is the elevation of the first line.

It is customary to use capital letters to denote points and lines, which are actually in the planes of reference, and to use small letters for the plan, and the same letters accented for the elevation, of other points.

Forces in Plan and Elevation.—Given the plan and elevation of a line and the projection in the elevation of a force acting in the line, to find the projection in plan.

In the figure, H V are the traces of the line, AH its plan, CV its elevation.





If $a'b'$ represents the projection of the force in elevation, ab will represent the projection in plan. For consider the parallelopiped OABD. The lines OA, OB represent the projections in the V.P. and H.P. of a force represented by OD.

Notice that the force cannot be replaced by the two projections T_{ab} and $T_{a'b'}$. The force is equivalent to $T_{a'b'}$ and a horizontal force represented by OC, for OD represents the resultant of OA and OC.

EXAMPLES.

41. Show that the force is equivalent to T_{ab} and a certain vertical *force.* *What* line in the elevation represents this vertical force?

42. Draw a "ground line" xy to represent the intersection of the horizontal and vertical planes. Take H 2 inches below xy, V $1\frac{1}{2}$ inches above xy and 2 inches to the right of H. The projection in the horizontal plane of a force acting in the line whose traces are H and V is 100 lbs. What is the projection in the vertical plane, and what is the magnitude of the force?

The method of solution applicable to the case of a number of forces meeting at a point in space, depends on the following propositions—

1. If any number of forces acting at a point are represented by the sides traversed consecutively of a closed non-coplanar polygon, commonly called a "skew" polygon, they are in equilibrium. The projection of such a polygon on any plane is a closed polygon, and the sides of this closed polygon represent the projections of the forces in that plane.

2. Conversely, if any number of forces acting at a point are in equilibrium there must be a closed skew polygon whose sides represent these forces, and the projection of this polygon on any plane is a closed polygon, and as its sides represent the projections of the forces on the plane the projections of the forces are in equilibrium (see p. 81).

3. If we can construct the projections of the forces on the vertical plane and also on the horizontal plane we can determine the magnitudes of the original forces.

4. If any number of forces act at a point in space, in given directions, and the magnitude of all but three of them are known, the magnitudes of these three can be found.

Perhaps the easiest way of seeing this is to write down *the three equations* expressing the fact that the sum of

the resolved parts of the forces along the axes Ox, Oy, Oz must vanish.

As the directions of all the forces are known, if the magnitudes of all but three are also known, the only unknown quantities in our equations will be these three unknown magnitudes; and three equations in general just suffice to determine three unknown quantities.

Graphically, we construct the resultant of the projections of all the known forces, in plan and in elevation. This reduces the system of forces in each plane to four, acting in known lines. We then construct the line of intersection of two planes, each plane containing two of the four forces.

We then construct a quadrilateral of forces in plan and elevation.

One method of proceeding, when the lines of action are given in plan and elevation, is indicated by the following example—

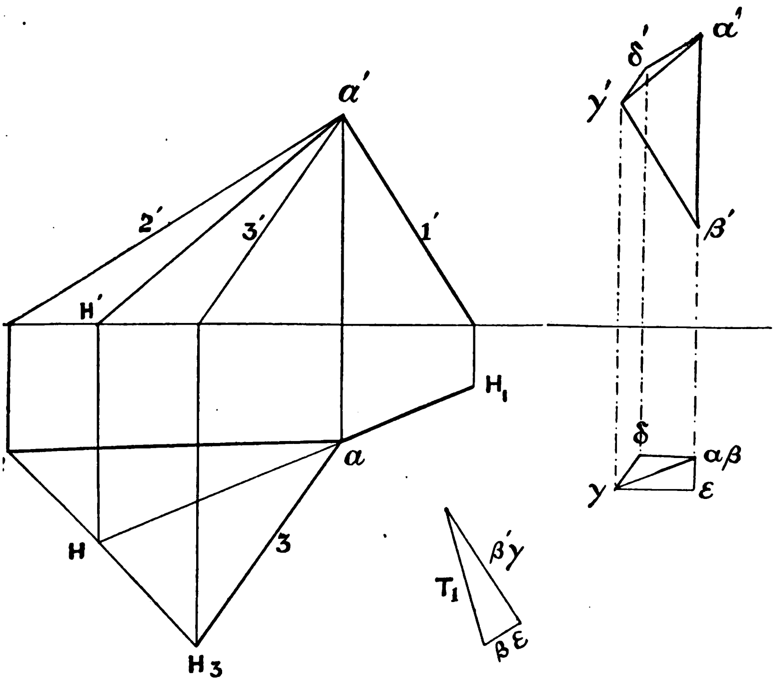
Example.

43. A three-legged tripod, shown in plan and elevation opposite, carries a load of 100 lbs. at its apex. Determine graphically the stresses on the three legs of the tripod.

In the diagram H_1 H_2, H_3 are the feet of the tripod legs, a is the plan, and a' the elevation of the apex. The four forces acting at the apex are the load W of 100 lbs. and the three stresses T_1, T_2, T_3 exerted by the three legs. The resultant of W and T_1 must act in the same line as the resultant of T_2 and T_3. But the resultant of T_2 and T_3 acts in the plane of the legs 2 and 3, and the

resultant of W and T_1 in the plane containing the vertical through a' and the leg 1.

Our first step is to find the line of intersection of these two planes. Now the horizontal trace of the plane through



2 and 3 is H_2H_3. The horizontal trace of the plane containing W and T_1 is aH_1.

Hence if H_2H_3 and aH_1 intersect in H then H is the horizontal trace of the line of intersection of these two planes. The elevation of this line of intersection goes through H′, the elevation of H, and through a'.

We have now sufficient data to construct a force polygon whose sides represent the projections in the vertical plane of the original forces, for the resultant of the projections of W_1 and T_1 acts in the line whose elevation is $H'a'$.

Commencing with the elevation, draw a vertical line $a'\beta'$ to represent 100 lbs. Draw lines $\beta'\gamma'$, $a'\gamma'$ parallel to $1'$ and to $H'a'$. Draw $\gamma'\delta'$ parallel to $3'$ and $\delta'a'$ parallel to $2'$. $a'\beta'\gamma'\delta'$ is the force polygon for the projections in elevation of the forces.

To construct the polygon, which in this case degenerates into a triangle, for the projections in plan we have only to determine points $a\beta\gamma\delta$ in plan.

Drawing the perpendicular $a'a$ and taking any point a on it (which is the plan of a' and also of β'), draw $a\gamma$ parallel to 1 and meeting the perpendicular from γ' in γ.

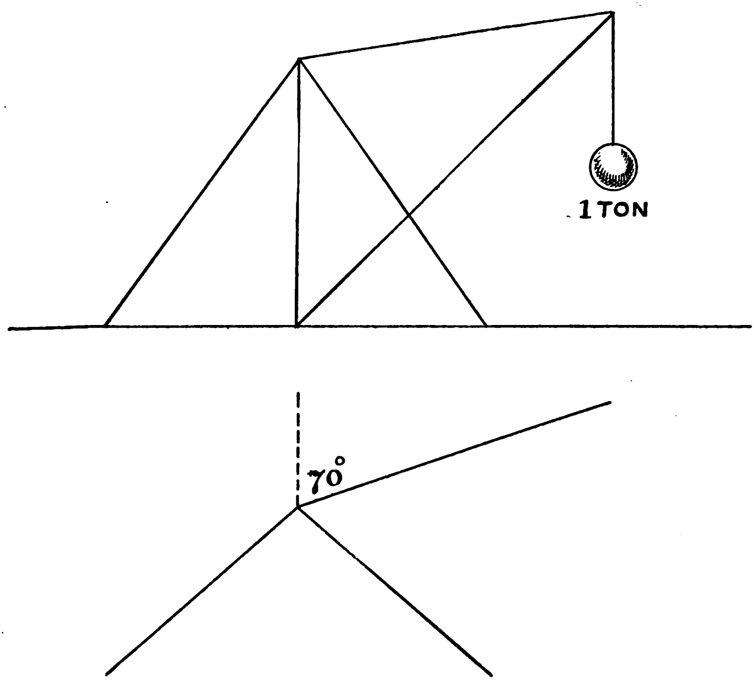
Draw $\gamma\delta$ parallel to 3, $a\delta$ parallel to 2. $a\gamma\delta$ is the required triangle. A check is furnished by the fact that $\delta'\delta$ should be perpendicular to the ground line. The actual stress in 1 is the resultant of forces represented by $\beta'\gamma'$ and the resolved part of $\beta\gamma$ perpendicular to the ground line. This resolved part is represented by the line $\beta\epsilon$, and thus the hypothenuse of a right-angled triangle whose sides are $\beta'\gamma'$ and $\beta\epsilon$ represents the stress in 1.

EXAMPLES.

44. If $H_1H_2H_3$ is an equilateral triangle, side 2 inches, H_2H_3 is perpendicular to the ground line, a is distant $\frac{1}{2}$ inch from H_1 and 1·8 inches from H_3, a' is $1\frac{1}{2}$ inches above xy, find the stresses in the three tripod legs. Find also the length of each leg.

45. The figure shows the plan and elevation of a derrick crane.

Find the traces of the line of intersection of the plane of the backstays with the plane of the crane itself. Hence determine the stresses in the five bars of the frame. Length of backstays, 22 feet;



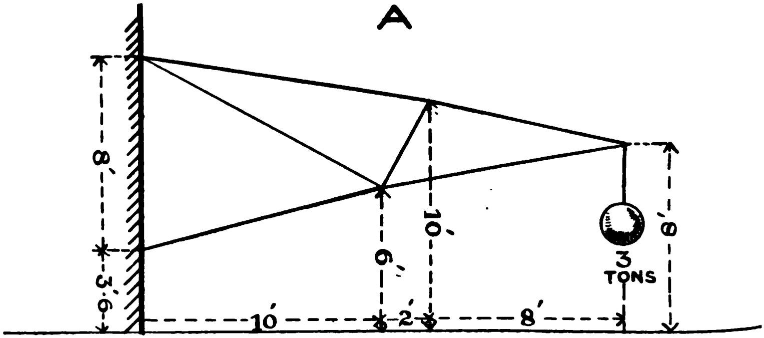
their feet 16 feet apart; height of vertical post, 15 feet; length of jib, 30 feet; length of tie-rod, 18 feet; load, 1 ton.

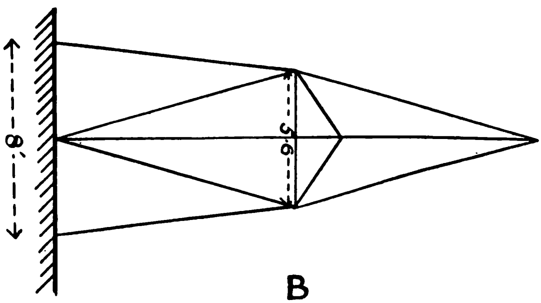
[Notice that we do, in effect, reduce the problem to one of coplanar forces. The resultant of the stresses in the two back legs must act in the plane of the crane itself.]

46. Suppose the crane to be in the same position as in Example 45 but to be pulling against a force of 1 ton acting in the plane of the jib and the tie-rod in a line whose elevation makes an angle of 45° with the vertical. Determine the stresses in the five bars.

47. A framework and its dimensions are shown in plan B and elevation A. Determine the stresses in the various bars.

A





B

48. The seat of a swing is a rectangle 24 × 18 inches, hung by four ropes, each 16 feet long, attached to its corners. The planes of the front and back ropes are vertical and the points of attachment of each pair of ropes 5 feet apart. A boy of 112 lbs. weight sits on the swing. Find the tension of each rope. If the seat is pulled forward by a horizontal force and held after being moved 6 feet horizontally, find the tensions in the ropes. Draw a plan and elevation.

MISCELLANEOUS EXAMPLES.

1. Explain the terms velocity, ratio, mechanical advantage and efficiency as applied to a machine. Illustrate by considering a system of pulleys in which a force of 10 lbs. weight is required to raise a load of 2 cwts., the former descending 2 feet while the latter rises 1 inch.

2. The beam of a balance whose arms are each 10 cm. long, and whose knife-edges are in one straight line, weighs 60 grms., and an excess of 2 mgr. in one scale-pan deflects the beam through an angle whose tangent is $\frac{1}{120}$. Prove that the centre of gravity of the beam is $\frac{2}{5}$ mm. from the point of support.

3. A picture 42 inches wide and 30 inches high is hung by two cords fastened to the vertical sides 7 inches from the top. Find the inclination at which it will hang if one cord breaks.

4. A man of 12 stones raises 3 cwts. by means of a system of pulleys, there being four in each block. Neglecting the weight of the pulleys and friction, find the pressure on the ground if he pull vertically.

5. What force applied at the end of an arm 18 inches long will produce a pressure of 1000 lbs. upon the head of a smooth screw, when eleven turns cause the head to advance $\frac{3}{5}$ inch?

6. In a block and tackle with three pulleys in the upper block and two in the lower, the weight of the whole being negligible, find the relation between the load and the force that would lift it if there were no friction.

If the friction is equivalent to one-tenth of the load, what force is required to lift 7 cwts.?

7. Make a sketch of an arrangement of pulleys in which the effort applied is multiplied by seven, neglecting all friction.

8. A system of pulleys has two movable pulleys A and B of weights 2 lbs. and 4 lbs. respectively, a fixed pulley C, and two strings, one passing over C with its ends attached to A and B, and

the other attached at one end to a fixed point, passing under A and over B, and having a mass of 3 lbs. fastened to its other end. The parts of the strings not in contact with the pulleys are all vertical. What weight must be attached to A to keep the system in equilibrium?

9. In a system of four pulleys consisting of a fixed block which contains two of them, and a movable block containing the others, the same rope passes continuously round all the pulleys. From the movable block is suspended a basket containing a man, the weight of man and basket being W. If another man standing on the ground pulls the free end of the rope, what force must he exert to raise the man and basket? Neglect friction.

If the free end of the rope is pulled by the man in the basket, what force must he exert for the same purpose? [Army.]

10. Can a man who is being weighed in a large pair of scales like an ordinary balance alter his apparent weight, by pressing a stick against the beam of the scales?

11. A straight uniform heavy lever, used as a common balance, has unequal arms of length 12 inches and $12\frac{1}{4}$ inches. A supposed pound is measured by hanging the body weighed from the *shorter* arm; and then two separate supposed pounds by hanging from the *shorter* and *longer* arms in turn. If G_1 is the customer's overweight in the first operation, and G_2 his net overweight in the pair of operations; prove that $G_1 = 49G_2$.

12. Two railway carriages are coupled by a right- and left-handed screw coupling. Given that the pitch of each of these is $\frac{1}{4}$ inch and the length of the tightening lever equal to 18 inches, find with what force the buffers are compressed if a pull of 50 lbs. be applied to the end of the lever. Neglect friction.

13. A man rolling a cask of 3 feet diameter and weight 294 lbs. pushes it up a plane inclined at 20° by a force applied at an upward inclination of 40° to a point 2 feet from the plane. Find the force and the work done in rolling the cask 20 feet up the slope.

14. If ABCD is a rectangle such that the diagonal AC is twice the side AB, and forces of 2 lbs., 4 lbs. and 8 lbs. act along AB, AC and BD respectively, show that their resultant is parallel to BC and find its position.

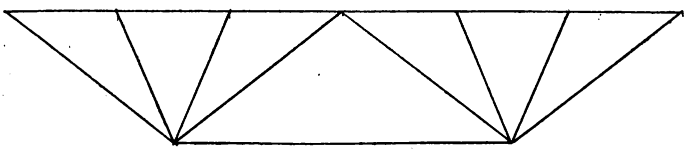
15. What is meant by (1) the arm of a couple; (2) the axis of a couple? Find the result of combining a couple of moment L with a force P acting in the plane of the couple.

16. In a vertical steam-engine the crank is 6 inches and the connecting rod 3 feet long. The horizontal crank-shaft is supported in bearings whose centres are situated at distances of 1 and 2 feet respectively on either side of the vertical centre line. At the instant when the crank is horizontal, the vertical thrust exerted by the piston is 2500 lbs. Find the magnitude of the turning couple on the crank-shaft and the direction and magnitudes of the thrusts on the bearings.

17. One smooth book, considered as a uniform rectangular block of height a and thickness b, leaning over at an angle α between books on either side of it on a smooth shelf will press against them with a force $\frac{W}{2} \cdot \frac{a \sin \alpha - b \cos \alpha}{a \cos \alpha - b \sin \alpha}$.

18. The sides of a rigid polygon are acted on by forces at right angles to the sides, and proportional to them in magnitude, all the force acting inwards on the plane of the polygon. Also the sides taken in the same order are divided by the points of application in the constant ratio $p : q$. Show that the system of force is equivalent to a couple of moment $\frac{\mu}{2} \frac{(p-q)}{(p+q)} \Sigma a^2$, where μa represents the force applied to any side of the polygon and Σa^2 is the sum of the square of the sides.

19. A truss of the form shown in the figure carries equal loads at each of the equally-spaced joints of the upper horizontal beam, and is supported by vertical reactions at its ends. The length of the horizontal tie-rod being half the span determine the ratio of depth to span in order that the pull in each tie-beam may be the same. Verify by drawing a reciprocal diagram.



20. ABCD is a rectangular door working in two vertical grooves AB, DC. The weight of the door is 0·72 ton, and its depth AB is 6 feet. The door is being raised by a vertical force of 0·8 ton in the plane of the door acting 9 inches from the centre of the door, and in

consequence tilting the door slightly in its grooves so that contact occurs only at A and C. Show that the reactions at these points are 0·1 ton, and calculate the coefficient of friction.

21. The total weight of a four-wheeled railway truck is 5 tons, and its centre of gravity is midway between the front and back axles, which are 10 feet apart. The drawbar is 3 feet above rail level. The axles are frictionless, but the brakes can lock either set of wheels. Calculate the pull required to move the truck, (*a*) with both wheels locked, (*b*) only front wheels locked, (*c*) only rear wheels locked. Take $\mu=0{\cdot}4$.

22. Two equal weights, W, rest on an inclined plane connected by a string lying along a line of greatest slope. If the coefficient of friction between the lower weight and the plane be $\frac{1}{3}$ and that between the upper weight and the plane $\frac{2}{3}$ and equilibrium be limiting, find the inclination of the plane and the tension in the string.

23. A ladder 50 feet long, weighing 120 lbs., whose centre of gravity is 20 feet from its lower end, rests with one end against a smooth vertical wall and the other on a rough horizontal plane at a distance of 14 feet from the foot of the wall.

Find the force of friction. Find also the coefficient of limiting friction, if suspending a weight of 20 lbs. from a point 40 feet up the ladder just produces slipping.

24. A weight Q of small dimensions is fixed upon the surface of a uniform sphere, of weight W. The sphere is placed on a rough board, 20 feet long, one end of which rests on the ground, the other end being supported by a prop 1 foot high. The sphere is in equilibrium, with the radius through Q inclined to the radius through the highest point at an angle of 30°. Find the ratio of Q to W, and the least possible value of the coefficient of friction between the sphere and the board.

25. A rod is guided so as to lift up and down freely in a vertical line. A wedge whose angle is α rests on a smooth horizontal surface and its thin edge is inserted beneath the lower extremity of the vertical rod. By means of a thrust applied horizontally to the wedge this lower end of the rod is made to rise in contact with the wedge along a line of greatest slope. Considering this arrangement as a machine for lifting the rod, and taking ϕ as the angle of friction between the rod and wedge, calculate (1) the mechanical advantage of the system, (2) its efficiency.

26. A weight is just sustained upon a rough inclined plane by a

force P acting along the plane, or by a force Q acting horizontally. Show that the weight is—

$$\frac{PQ}{\sqrt{Q^2 \sec^2 \alpha - P^2}},$$

where α is the angle of friction.

27. A railway train weighing 100 tons is drawn up an incline of 1 in 40 by a rope. What is the least pull on the rope if 10 lbs. per ton be allowed for frictional resistance to motion?

28. A uniform cylinder is placed, so that its axis is horizontal, upon a rough inclined plane and kept in position by a string which is led off horizontally from the cylinder and fastened to the plane. The string is in the vertical plane, which is perpendicular to the axis of the cylinder. Show that, if the cylinder is in limiting equilibrium, the coefficient of friction must be equal to

$$\tan \frac{\alpha}{2},$$

α being the angle of inclination of the inclined plane.

29. A uniform rod rests inside a rough circular hoop, the upper end being at the end of a horizontal diameter in the position of equilibrium. If α is the inclination of the rod and μ the coefficient of friction between ends of rod and hoop, prove—

$$\tan^3 \alpha - \mu^2 \tan \alpha = \mu\,(1 + \tan^2 \alpha.)$$

30. Prove that if ϕ is the maximum incline on which a four-wheeled carriage can stand when both pairs of wheels are locked, and θ the maximum incline when only one pair is locked, then $\tan\,\theta = \frac{l}{l + m} \tan \phi$, if in the latter case the vertical through the centre of gravity of the carriage cuts the road in a point which divides the distance between the wheels in the ratio $l : m$.

31. A reel of cotton, radius of ends a, radius of central portion b, is placed on a plane of inclination α so that the cotton would unwind as the reel rolls down the plane. The cotton leaves the reel parallel to the inclined plane. If μ is the coefficient of friction, show that to hold the reel in its place μ must be not less than $\frac{a}{a-b} \cdot \tan\,\alpha$. Examine the result if the thread leaves the reel horizontally instead of parallel to the inclined plane.

32. A plane is inclined at an angle α to the horizon, the upper

edge being horizontal. A uniform chain of length l is hung over the top. Find the greatest and least lengths of the vertical portion if the angle of friction between the plane and the chain is λ. As a particular case take $l = 2$ feet, $a = 50°$ and $\lambda = 20°$.

33. A cylindrical cask 3 feet in diameter weighing 200 lbs. is being rolled up a plank incline by means of a small crowbar 4 feet long. If the incline makes an angle of 15° with the horizontal, find the pull which must be exerted at the end of the bar when the bar is perpendicular to the incline.

34. On a wheel and axle, the bearings of the axle being rough, P is the least downward power which will support a weight W and P′ the greatest downward power which can be applied downwards without lifting W. Prove that

$$(P+W)(P'a-Wb) + (P'+W)(Pa-Wb)=0,$$

where a is the radius of wheel and b of the axle.

35. A straight lever ABC, whose weight need not be considered, rests with its end C on a rough horizontal floor, and its middle point B is pressed against a smooth support by a horizontal force applied at the top end A. Show that C will slip either forward or backward unless the angle of inclination of the lever to the horizontal lies between

$$\frac{1}{2}\tan^{-1}\left(\frac{1}{\mu}\right) \text{ and } \frac{1}{2}\tan^{-1}\left(-\frac{1}{\mu}\right),$$

μ being the coefficient of friction.

Draw diagrams showing the forces acting on the lever in each of its limiting positions, taking $\mu=0{\cdot}5$.

36. Two rough planes of the same material are at right angles, and one of them makes an angle a with the horizon. A heavy rod resting upon them is about to slip, show that the normal pressures on the planes are—

$$W \cos\lambda \cos(a-\lambda) \text{ and } W \cos\lambda \sin(a-\lambda),$$

where $\tan\lambda$ is the coefficient of friction.

37. Two children, each of weight W, are swinging on a see-saw, formed by placing a plank, of weight w, across a horizontal cylinder of radius c.

When the plank has been turned through a given angle, find the moment of the forces tending to turn it back.

Also, prove that the greatest angle through which the plank can

swing without slipping is double of the angle of friction between the plank and cylinder.

38. A heavy body is supported on a smooth inclined plane by a force whose direction makes an angle β with the plane. Find the force, and show that the pressure on the plane is—

$$W\frac{\cos(\alpha \pm \beta)}{\cos\beta},$$

where W is the weight and α the inclination of the plane to the horizon. Explain the double sign.

Also, taking the upper sign, explain the cases where—

$$(1)\ \alpha + \beta = 90^\circ;\ (2)\ \alpha + \beta > 90^\circ.$$

39. A uniform heavy rod, $6\frac{1}{2}$ inches in length, is suspended from a point by two strings 6 inches and $2\frac{1}{2}$ inches long respectively. Prove that, in the position of equilibrium, the rod makes an angle $2\tan^{-1}\frac{5}{12}$ with the vertical.

40. A uniform window 7 feet in height weighing 40 lbs. is pivoted $2\frac{1}{2}$ feet from the top. It is held open in a horizontal position by a cord which connects the top of the window with a point 6 feet vertically below the pivots. Find, by a graphical method or otherwise, the tension of the cord and the thrust on the pivots.

41. A two-wheeled carriage, of weight W, has to be drawn over an obstacle of height h, which lies across the road, by means of a pull applied by horizontal traces at the level of the axles. Show that this pull must be at the least of magnitude

$$\frac{(2ah - h^2)^{\frac{1}{2}}}{a - h}W,$$

a being the radius of the wheels, in order to surmount the obstacle.

42. If a uniform heavy rod hangs from a peg by two strings, whose weight is neglected, each equal in length to the rod, find the tension of the strings.

If the strings are replaced by uniform heavy rods, like the original rod, freely jointed at all the angular points, find the direction and magnitude of the resultant action at the lower joints.

43. A uniform rod is placed between two equally rough planes, one of which is vertical and the other inclined at θ with it, and just rests where it makes equal angles with both, prove that—

$$\sin^2\frac{\theta}{2} - \mu^2\cos^2\frac{\theta}{2} = \mu\cot\frac{\theta}{2}.$$

44. A uniform rod rests over a smooth fixed sphere, one end pressing against a smooth vertical plane which touches the sphere; if θ is the angle which the rod makes with the vertical in the position of equilibrium, l its length, and a the radius of the sphere, prove—

$$a = 2l \sin \frac{\theta}{2} \cos^3 \frac{\theta}{2}.$$

45. A uniform beam AB turning about a hinge at A touches a smooth hemispherical surface at B. The base of the hemisphere is rough and rests on a horizontal plane passing through A, the coefficient of friction being μ. If W = weight of hemisphere and w that of the beam, find the greatest weight that can be placed at the centre of the beam without producing motion, the length of beam, and radius and position of hemisphere being given.

46. Two fixed bars in a vertical plane are each inclined at 45° to the vertical. The ends of a string are tied to rings each of weight W_1, which slide without friction on the bars. From the string a weight W_2 is hung. Prove that each half of the string will make an angle θ with the vertical given by

$$\tan \theta = 1 + \frac{2W_1}{W_2}.$$

47. A string 6 inches in length has one end tied to a fixed point and the other end to a small ring B. A second string is attached to a point in the same horizontal line as A and 10 inches from it—passes through the ring B and supports a weight of 11 lbs. Prove that if θ is the inclination of the first string to the horizontal

$$10 \cos^2 \theta - 3 \cos \theta - 5 = 0.$$

48. A uniform beam ABC is placed with one end A on a smooth fixed hemispherical bowl whose rim is in a plane inclined at $a°$; the beam is longer than the diameter of the bowl and rests upon the rim. If r is the radius of the bowl and $2a$ the length of the beam, and θ the angle of inclination to the horizontal, show that

$$2r \text{ co } (2\theta \pm a) = a \cos \theta.$$

49. A uniform heavy plank AB, length $2a$, weight W, is held at a given inclination a to the vertical by a rope attached to the end *B, and passing* over a pulley C. The pulley is at a height of h feet

and vertically over the lower end A of the plank, which rests on the ground. Find the tension of the rope.

50. (1) A pair of equal uniform heavy rectangular laminas are freely jointed along corresponding sides, and stand with their opposite sides resting each on one or two smooth planes which are each inclined at an angle α to the horizon. Find the inclination of each lamina to the vertical.

(2) If instead of the two smooth inclined planes we have one rough horizontal plane, find the limiting position of equilibrium.

51. A circular plate of radius a feet is fixed in a vertical plane. Upon it (and in its plane) are placed two uniform heavy rods AB, BC, jointed at B, each being $2b$ feet in length and weighing W lbs. Find an equation for θ, the angle which either rod makes with the vertical in the position of equilibrium, and express the pressure upon the plate and the reaction at the joint in terms of W and θ. Friction is to be neglected. [Army]

52. A uniform rectangular plate ABCD is free to turn in a vertical plane about a hinge at A, and the plate is held in a position in which D is above and B is below the level of A by a horizontal force equal to the weight of the plate applied at a point P of AB. Prove that

$$\text{AP} = \tfrac{1}{2}(\text{AD} + \text{AB} \tan \alpha),$$

where α is the inclination of AD to the horizontal. [Army]

53. A ditch is bridged over by a light plank AB, supported at its two ends. A man of weight W stands at any point P of the plank.

(i) Find what portions of his weight are supported by the reactions at the two ends A and B respectively.

(ii) Take any point Q between P and B, and prove that the moment about Q of the forces acting on either one of the portions AQ or QB is

$$\text{W}.\frac{\text{AP}.\text{QB}}{\text{AB}}.$$

(iii) If this moment measures the tendency of the plank to break at Q, prove that the plank will be most likely to break in the middle when the man is standing there. [Army]

54. Two uniform rods each 26 feet long weighing 60 and 30 lbs. respectively are hinged together at their upper extremities and stand

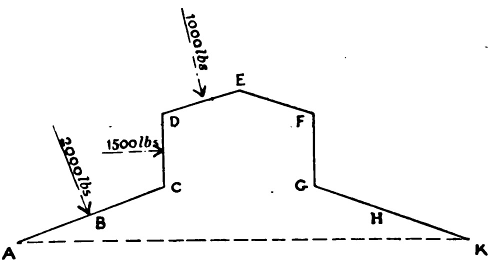
on a smooth horizontal plane, being kept in position by a string 10 feet long joining their middle points. Find the tension of this string and the reaction at the hinge.

55. A uniform hoop weighing 2 lbs. rests horizontally on pegs A B C; the arcs AB, BC, CA subtending 130°, 100° and 130° at the centre. A weight of 14 lbs. is tied by strings AD and ED 24 and 30 inches long to A and to the mid point E of the arc BC. Find the pressures on the pegs.

56. A camp stool, whose legs are pinned together at their middle points, and are each 2 feet 6 inches long, and the canvas seat 18 inches across, has a smooth weightless board placed symmetrically upon it, and a weight of 100 lbs. placed on the middle of the board. Find the tension of the canvas.

57. Two equal beams ABC, BCD, each 10 feet long, are placed side by side, overlapping by the length BC, namely, 4 feet. They are lashed together at B and C and placed with the ends A and D resting on two supports in the same horizontal line. Each beam weighs 200 lbs., and a load of 1200 lbs. rests on the middle part of AB. Assuming the beams to be in contact only at B and C, find the reactions at B and C and the pressures on the supports.

58. A uniform rod ACB rests against a smooth floor at A and a smooth wall at B, and is prevented from slipping by a rope tied to C and to the foot of the wall. Calculate the tension of the rope in terms of AC, AB, and the weight of the rod. Examine the cases (*a*) AC greater than half AB, (*b*) AC very nearly equal to half AB.



59. The outline of a roof is shown in the diagram. Loads of 600 lbs. act at each of the points BCDFGH, and at A and K are loads of 300 lbs. The forces indicated in the diagram arise from wind

pressures. Find the resultant of all the applied forces other than the reactions at A and K.

If the reaction at K is vertical, find it, and the reaction at A.

60. Show that in general a given force may be decomposed in one and only one way into three forces acting in given lines in its plane Examine the exceptional cases.

61. Show how, by means of a link polygon, to replace a given force acting in a given line by three forces along three given non-parallel straight lines.

When is the problem impossible?

62. At ABC are weights W_1, W_2, W_3. G is their centre of gravity and O any point. Take OG as axis of x. From the equation—

$$AG^2 = AO^2 + OG^2 - 2AO \cdot OG \cos AOG,$$

and similar equations for B and C, deduce the equation—

$$W_1AG^2 + W_2BG^2 + W_3CG^2$$
$$= W_1AO^2 + W_2BO^2 + W_3CO^2 + (W_1 + W_2 + W_3)GO^2.$$

Hence show that if the sum of the squares of the distances of a point from the angular points of a triangle is to be as small as possible, that point must be the centre of gravity of the triangle.

63. A uniform very thin lamina AOB is in the shape of a quadrant of a circle, centre O, radius $OA = OB = a$. Given that its centre of gravity G is on the bisector of the angle AOB, and that

$$OG = \frac{4\sqrt{2}}{3\pi} a,$$

find the distance from O of the centre of gravity of the portion of the lamina bounded by the curve and the chord AB, and show that if $a = 17$ inches this distance is 14 inches nearly.

64. A rectangular slab of stone, ABCD, stands upright on a horizontal plane. A triangular piece, ABM, is removed, M being a point in the base BC, and the remainder of the stone just does not upset. Find the ratio of BM to BC.

65. A triangular lamina 1 inch thick, sides 15, 15 and 24 inches long, weighs 70 lbs. It is placed upright with one of its shorter sides on a horizontal plane. Find what weight can be suspended from the vertex without upsetting the triangle.

66. A uniform heavy triangular lamina ABC is freely suspended at A. Determine the position of rest.

If weights equal to the weight of the lamina are hung at B and C, show that the position of equilibrium is the same as if the portion GBAC had lost its weight, G being the centre of gravity of the lamina.

67. Find the centre of gravity of a portion of a body, given the weight and the centre of gravity of the remainder of the body and of the whole body.

A circle of radius r touches another circle of radius r' internally at B. Prove that the centre of mass of a disc whose form is that of the figure enclosed between the two circles is at a distance $(r^2 + rr' + r'^2) \div (r + r')$ from the point B.

68. If any number of forces in one plane acting on a rigid body have a resultant, the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant.

Forces respectively proportional to $\cos B - \cos C$, $\cos C - \cos A$, $\cos A - \cos B$, act along the sides BC, CA, AB of a triangle ABC. Prove that the line of action of their resultant passes through the centres of the inscribed and circumscribed circles.

69. A piece of paper is in the form of a square ABCD (uniform in thickness and material); O is the centre of the square, and E, F are the middle points of the sides AB, AD respectively. The paper is folded across EF so that A coincides with O. Find the centre of gravity of the paper when so folded.

70. O is any point inside a triangle ABC; show that forces along OA, OB, OC whose magnitudes are proportional to OA × area of the triangle BOC, OB × area of COA, OC × area of AOB respectively, are in equilibrium.

71. A uniform plate is in the form of a rhombus, and one corner formed by joining the middle points of two adjacent sides is cut off. The plate is then suspended from a vertex. Show that the angle which the diagonal through the vertex makes with the horizontal has its tangent equal to $\dfrac{21d'}{2d}$, where d, d' are the lengths of the diagonals.

72. At the angular points of a square are placed equal particles of the same attracting matter. If the force of attraction varies as the distance, show that the resultant attraction at all points on the circumscribing circle is the same.

73. A rectangular trap-door ABCD, weighing 100 lbs., is

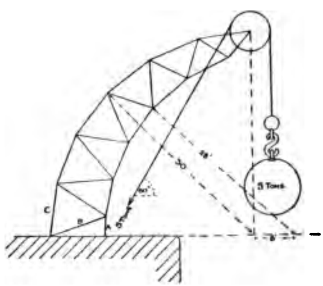
supported in a horizontal position by two hinges at EF, two points in BC, and by a string attached to A. The string is fastened to a nail vertically above B, and is inclined at 45° to the horizontal. Draw diagrams showing (1) the vertical forces acting on the door, (2) the forces parallel to AB acting on the door. Hence find the reactions at the hinges, excluding any force there may be between the hinges in the direction of the line joining them. AB = 2 feet, BC = 4 feet, BE = FC = 1·5 feet.

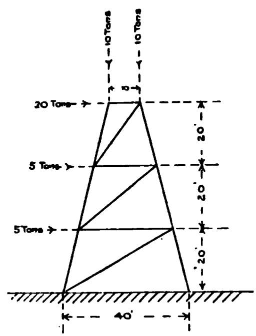
74. A string AB fastened to a fixed point A has a weight of 25 lbs. tied to the point B. Another string, BC, knotted to the first at B passes over a pulley and carries a weight of 11 lbs. Given AB = 24 inches, AC horizontal = 36 inches, find the tension in AB and the depth of B below AC.

75. Assume that the pressure of wind against a fixed plane acts at right angles to the plane and is proportional to the cosine of the angle which the wind makes with the normal to the plane. Find the position of the plane in order that the thrust on it, in a direction perpendicular to the wind, may be a maximum.

[This result shows how the sails of a wind-mill should be set.]

76. The bracing of the crane shown in the accompanying figure consists of isosceles triangles having equal bases upon the outside arc,

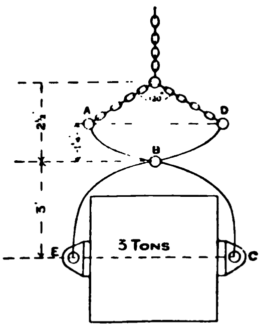




whose radius is 30 feet. The radius of the inner arc is 28 feet, and the bars A and C are 8 feet apart at the ground level. Determine graphically the stresses in the bars A, B and C when the crane is supporting a 5-ton load in the manner shown.

77. The loads and dimensions of a braced pier are shown in the right-hand figure on p. 371. Construct Bow's figure for the framework and draw up a table showing the stresses in the various bars.

78. The accompanying figure represents a grab used for lifting stone blocks. The flat grips E and C are pressed against the side of the block by levers ABC, DBE freely hinged at B, and if the mechanism is suitably designed, the friction between the grips and the block is sufficient to lift the block. The block shown is a cube having an edge of 4 feet. Show that if the coefficient of friction between the grips and the block is $\frac{1}{2}$ the grab will be able to lift the block. Calculate also the least coefficient of friction for which this is possible.



79. A sliding door rests on two wheels which run on a horizontal rail. The centre of gravity of the door is 5 feet above the rail level and the two wheels are spaced 3 feet on either side of the vertical through the centre of gravity.

The door is pushed forward by a horizontal force applied at a height of 4 feet above the rail level. The first wheel is seized in its bearings and refuses to turn, and the other rotates freely. Calculate the force required to move the door. The weight of the door is 800 lbs. and the coefficient of friction for the wheel sliding on the rail is 0·2.

Xd 3'

JW